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Foster Student Understanding of Parallelism Based on the Toulmin Model and Pirie-Kieren Theory

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ABSTRACT: Students' understanding of the definition and classification of quadrilaterals serves as a crucial foundation in geometry learning because it allows students to make connections between geometric figures based on their essential properties. Previous studies have primarily examined the structure of arguments or levels of geometric thinking, leaving the dynamics of the development of understanding during the argumentation process inadequately explored. This study aims to how collective argumentation fosters the development of junior high school students' understanding of the definition and classification of quadrilaterals through collective argumentation. The research framework integrates Toulmin's argumentation model to analyze the structure of arguments with Pirie and Kieren's theory of the development of understanding to identify how it fosters trajectories of understanding. This study employs a qualitative approach involving a mathematics teacher and eighth-grade students selected through purposive sampling. Data were collected through classroom recordings, student work samples, observations, field notes, and interviews. The results indicate that the development of understanding progresses through the layers of primitive knowing, image making, image having, property noticing, formalising, and observing, accompanied by the process of folding back, while the characteristics of structuring were not fully and explicitly identified in the data. The integration of these two theories shows that the Toulmin model reveals the construction of arguments, while the Pirie-Kieren theory explains the development of understanding that underlies those arguments. These findings emphasize that collective argumentation fosters a mechanism that drives the reorganization of knowledge and strengthens students' conceptual understanding of the definition and classification of quadrilaterals.

KEYWORDS: Foster Student Understanding; Toulmin Model; Pirie-Kieren Theory

I. INTRODUCTION

Mathematics instruction aims not only to help students understand concepts but also to foster the development of understanding that allows for the formation of increasingly integrated connections between concepts. This development of understanding enables students to make connections between new knowledge and prior knowledge, thereby allowing them to apply these concepts in various problem-solving situations (Alreshidi, 2023). In line with Hiebert and Carpenter (1994), understanding is demonstrated through the connections between a person's existing knowledge. Therefore, the development of understanding is reflected when students become more capable of explaining concepts in their own words, distinguishing between examples and non-examples, and providing reasons for the procedures used to solve a problem.

An understanding of mathematical concepts is essential across all topics studied by students, including geometry. One area of geometry that requires a deep conceptual understanding is the definition and classification of quadrilaterals (Sprenger, 2026). In this subject, students must not only recognize the shape of a geometric figure but also understand the connections between the properties that form the basis for classifying various types of quadrilaterals. A key concept in this process is parallelism. The concept of parallelism serves as the foundation for defining the family of parallelograms, including parallelograms, rectangles, rhombuses, and squares (Birgin & Özkan, 2024). By understanding the concept of parallelism, students are expected to be able to explain the relationships between geometric figures logically, rather than merely recognizing their figures visually (Fujita & Jones, 2014).

Parallelism does not merely cover the properties of plane figures, but is a concept that defines the relationships among figures within the family of quadrilaterals. The presence of two pairs of parallel sides is the characteristic that distinguishes the family of parallelograms from other groups of quadrilaterals (Zembat & Gürhan, 2023). The understanding of this concept allows students to explain why squares, rectangles, and rhombuses are classified as special cases of parallelograms based on their properties, rather than merely because of their visual similarity. This understanding serves as the foundation for students to formulate a definition, classify quadrilaterals hierarchically, and construct mathematical justifications that support the relationships among figures (Fujita et al., 2019; Fujita & Jones, 2008). Students generally recognize rectangles or parallelograms based on the shapes most frequently presented in textbooks. As a result, when the orientation of a shape is changed or relationships between properties need to be considered, students tend to rely on their existing visual images rather than using formal definitions. This indicates that students' understanding is still dominated by the concept image, while the concept definition has not yet developed sufficiently (Tirosh & Tsamir, 2022).

One method that can foster this understanding is through learning that provides opportunities for argumentation. When expressing their opinions, students not only present their answers but also explain their reasoning, use the information they have to support their claims, and respond to their peers' opinions during the discussion process (Conner, 2022). Through these activities, students learn to connect their prior knowledge with new information, allowing them to develop a more meaningful understanding. In line with Conner et al. (2014), argumentation serves not only as a form of communication but also as a way to demonstrate how students construct and develop their mathematical understanding. Through argumentation, students learn to be accountable for the ideas they present. When the reasons they provide are challenged by their peers or teachers, students need to reevaluate their existing knowledge, revise any inaccurate reasoning, and formulate more logical justifications. This process enables the restructuring of knowledge, thereby making argumentation not only a form of communication but also a mechanism for the fostering understanding.

This argumentation process needs to be analyzed using a framework capable of illustrating how students construct their understanding. One of the frameworks widely used in mathematics education research is the Toulmin argumentation model. Through this model, students' thought processes can be explored from the way they use the information they possess, construct arguments, to drawing a conclusion (Miller et al., 2023). Argumentation analysis not only determines whether students' answers are correct or incorrect but also provides insight into the reasoning process underlying the formation of their understanding (Toulmin, 2003).

Argumentative structure can illustrate how students formulate a conclusion, but it does not fully account for the development of understanding that occurs during the argumentation process (Reuter, 2023). Students may produce similar argumentative structures yet demonstrate varying levels of understanding. This suggests that analyzing argumentative structure alone is insufficient to capture the dynamics of students' understanding. To explain these dynamics, this study employs Pirie and Kieren's (1994) theory of the development of understanding, which views understanding as a dynamic process involving multiple layers of understanding. In this process, students can expand their understanding, draw on prior knowledge (folding back), and develop a deeper understanding (George & Voutsina, 2024). Thus, Pirie and Kieren's theory complements the Toulmin argumentation model by explaining how understanding develops during the argumentation process.

Combining the analysis of understanding development and argumentation provides a more comprehensive overview of the mathematics learning process. Analysis of understanding development allows researchers to identify how students construct, connect, and reorganize mathematical concepts during learning, while analysis of argumentation helps explain how the processes of reasoning, providing justifications, and responding to others' opinions contribute to that development (Pirie & Kieren, 1994; Toulmin, 2003). The development of students' understanding can be observed not only through changes in the layers of their understanding but also explained through the accompanying argumentation process.

Research on the definition and classification of quadrilaterals has largely relied on Van Hiele's theory to identify students' levels of geometric thinking (Fujita & Jones, 2008). These studies provide an overview of the level of understanding students have achieved, but they do not yet explain in detail how that understanding develops during the learning process. On the other hand, research on mathematical argumentation generally focuses on the structure of students' arguments, whereas research on the development of understanding has largely relied on Pirie and Kieren's theory without linking it to the argumentation process. Therefore, research that integrates Toulmin's argumentation model with Pirie and Kieren's theory to examine the development of students' understanding of the definition and classification of quadrilaterals remains limited. The integration of these two frameworks constitutes the contribution of this study in providing a more comprehensive overview of how argumentation reflects the development of students' understanding.

Based on the above discussion, this study aims to describe how collective argumentation can foster the development of students' understanding of the definition and classification of quadrilaterals through an analysis of their arguments based on the Toulmin model and the theory of Pirie and Kieren. The findings of this study are expected to enrich the literature on mathematical argumentation and the development of understanding in geometry instruction. In addition, the findings of this research are expected to provide input for teachers in designing lessons that encourage students to foster conceptual understanding through mathematical argumentation.

II. THEROTICAL FRAMEWORK

This study employed two theoretical concepts that are used consistently: (1) the complete Toulmin argumentation model as a tool for mapping the structure of mathematical arguments; and (2) the Pirie & Kieren model as a description of the developmental trajectory of understanding that emphasizes the shift from intuition to justification through the negotiation of social meaning in the classroom. The Toulmin model analysis was applied comprehensively, starting from (D) data; (C) claim; (W) warrant; to (B) backing, and also included (Q) qualification to comprehensively document the quality of the argument structure within this framework, as well as (R) rebuttal. This study employed a qualitative research design to describe how teacher-supported collective argumentation fosters the development of junior high school students' understanding of the definition and classification of quadrilaterals. The students were divided into groups of three. Subject groups were selected purposively based on information provided by the teacher, taking into account the students' mathematical ability, communication skills, and engagement during discussions so that the process of fostering understanding could be observed in depth using the theory proposed by Pirie and Kieren (1994).

Table 1. Layers of Understanding in Pirie and Kieren's Theory

Layers of Understanding	**Description**	**Anticipated Student Activities**
Primitive knowing	The initial layer in which students use their prior knowledge to understand new concepts.	1. Understand the necessary concepts, how to formulate definitions, and the information being asked for. 2. Identify sides, angles, diagonals, opposite angles, acute and obtuse angles, parallel lines, the relationship between parallel lines intersected by a third line, and the conditions for congruent triangles.
Image making	The layer in which students construct representations of concepts based on their prior knowledge and observed examples.	1. Draw a parallelogram to scale. 2. Measure sides, angles, or diagonals. 3. Making conjectures about the congruence of sides, angles, and diagonals. 4. Formulating definitions based on drawings or examples.
Image having	The layer in which students have a mental image of the new knowledge, but are no longer tied to specific examples.	1. Drawing a parallelogram using symbols without measurements. 2. Demonstrating congruence through symbols. 3. Formulating definitions based on mental images, rather than on examples or models. 4. Presenting a description of a parallelogram along with its properties.
Property noticing	The layer in which students are able to identify the properties of concepts that were reconstructed in the previous layer.	1. Identify the properties of opposite sides, opposite angles, and diagonals in a parallelogram.
Formalising	The layer in which students construct definitions or concepts based on the properties they have identified.	1. Develop patterns of shared properties. 2. Identify shared properties among geometric figures. 3. Classify parallelograms based on shared properties.
Observing	The layer in which students organize the results from the formalising layer to solve related problems they encounter.	1. Connect quadrilateral figures based on shared properties. 2. Construct a hierarchical diagram of the quadrilateral family.
Structuring	The layer in which students are able to relate one theorem to another and prove them using logical arguments.	1. Relate the concepts of parallelism and congruence to the concept of a parallelogram. 2. Construct a proof that the opposite sides of a parallelogram are equal in length using definitions or relationships between properties.
Inventising	The layer in which students have a complete framework of understanding and can apply previous concepts to new ones.	1. Create new relationships or structures among quadrilaterals through different definitions.

Research data were collected through a learning activity involving comprehension tasks on the definition and classification of quadrilaterals, which were completed through collective reasoning in small groups. During the discussions, the teacher provided support as needed to foster the development of students' reasoning. Research data were obtained from video recordings of the learning sessions, students' work, teacher and student observation sheets, field notes, and interviews. The research instruments, consisting of comprehension worksheets, observation sheets, and interview guidelines, were validated by experts before being used in the study.

The data analysis referred to the model by Miles et al. (2020), which includes data condensation, data presentation, and the drawing and verification of conclusions. Video recordings of lessons, interview transcripts, and student assignments were first transcribed and then analyzed in stages. The analysis focused on the process of students' understanding development based on Pirie and Kieren's theory by identifying the characteristics of each layer of understanding that emerged during collective argumentation. The structure of students' argumentation was used as the basis for interpreting the dynamics of understanding development, while teacher support was analyzed as part of the learning context that fostered this process. The interpretation of each layer of understanding was conducted iteratively through triangulation of classroom observations, student assignments, and interview data, so that the characteristics of each layer were determined based on the consistency across various data sources.

III. RESULT AND DISCUSSION

A. *Fostering Students' Understanding trough Argumentation*

The research findings indicated that the development of students' understanding of the definition and classification of quadrilaterals was gradually fostered through collective argumentation. This development was evident as students constructed the definition of a quadrilateral, classified geometric figures based on their properties, and came to understand the relationships among figures. At each layer, students demonstrated shifts in their thinking fostered by the discussion process and teacher support. The research findings were further elaborated based on these three activities.

1. *Primitive Knowing*

The development of students' understanding began at the primitive knowing layer, when students start defining a parallelogram based on their prior knowledge. At this layer, students proposed various ideas regarding the properties of a parallelogram based on their previous learning experiences and began to synthesize them into a definition.

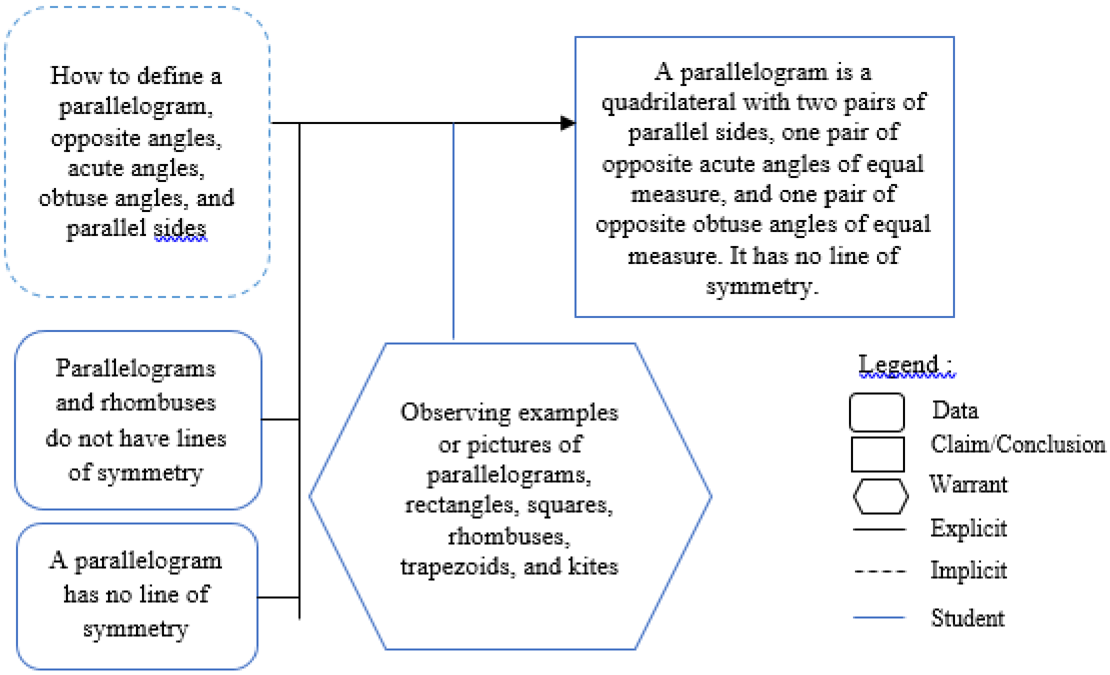

Figure 1. Sub-arguments for Defining

The students' argumentation began with an attempt to recall the basic properties of a parallelogram that they had previously learned. This was evident when the students stated that a parallelogram is "a plane figure with 2 pairs of sides," and then refined this to "a plane figure with 4 sides." The students then noted that a parallelogram "is a plane figure … that has one … two pairs of opposite sides of equal length." They also pointed out that "opposite angles are equal in measure" and added that "angles C and D are both acute, while angles A and B are both obtuse" as properties they considered to support this definition. These various statements indicated that the students were able to identify the key attributes of a parallelogram based on their prior knowledge. In addition to recalling these properties, the students also compared several examples of quadrilaterals to verify that the properties matched before agreeing on a definition of a parallelogram. The schema demonstrated that the conclusions drawn by the students still relied on their observations of geometric examples and the prior knowledge they were able to recall, thus the logical relationship linking these properties to the formal definition of a parallelogram had not yet been fully articulated.

These characteristics of the argumentation suggested that students still relied on prior knowledge as the foundation for fostering their understanding. Although the properties of a parallelogram were successfully identified, they were still viewed as isolated pieces of information and had not yet been connected into a coherent definition. This situation indicated that students had not yet explained why these properties are sufficient to define a parallelogram or to distinguish it from other quadrilaterals. Thus, students' thinking process was still dominated by recognizing figures and recalling previously learned characteristics rather than formally establishing conceptual relationships among these properties.

From the perspective of Pirie and Kieren's theory, this activity exhibited the characteristics of the primitive knowing layer. At this layer, understanding began with the activation of prior knowledge, which was still in the form of fragments of information, before developing into a more organized conceptual structure. The differences in students' knowledge then became a source of collective argumentation, allowing them to complement one another's ideas and negotiate them to foster shared understanding. This finding aligned with Martin and Towers (2009), who stated that individual knowledge in the early layers was represented through separate pieces of primitive knowledge that began to connect with one another through shared mathematical activities.

2. Image Making

Students' understanding continued to develop at the image making layer, when they began to form a mental picture of the concept of a parallelogram through exploration of its properties. At this layer, students no longer merely recognized the figure of a parallelogram visually but began to connect that figure with its defining characteristics, such as pairs of parallel sides, sides of equal length, and opposite angles of equal measure. This process indicated that students were beginning to foster a connection between visual representations and their formal conceptual understanding of parallelograms.

Representations of the concept of a parallelogram were formed through the process of selecting, combining, and simplifying various properties considered to characterize the figure. This was evident when students stated that a parallelogram "does not have fold symmetry," then added "because a parallelogram has two intersecting diagonals of equal length," and reaffirmed "it has two intersecting diagonals of equal length." Moreover, students attempted to formulate a more concise representation by stating, "To put it simply, it has two pairs of sides that are equal in length and parallel," then adding, "two pairs of opposite angles that are equal in measure," and finally concluding, "it has two intersecting diagonals of equal length and lacks fold symmetry." This sequence of argumentation demonstrated that students no longer merely listed the properties of a parallelogram separately but began to integrate various attributes into a single representation that they believed captured the concept of a parallelogram. Nevertheless, the reasoning behind the selection of these attributes was still influenced by the characteristics visible in the visual representation, so some of the included attributes were not entirely essential in defining a parallelogram.

This activity demonstrated the characteristics of the image making layer, in which students fostered their understanding through representations that helped them form mental images of a concept. Unlike the primitive knowing layer, which involved recognizing attributes separately, students at this layer began to connect several properties into a single unified representation. However, the representations they constructed were still influenced by the geometric figures they observed, so students were not yet fully able to distinguish between essential attributes and those that appeared only in specific representations. This indicated that students' understanding began to develop but still relied on visual experience in constructing concepts.

These findings were consistent with Alcock and Simpson (2017), who stated that concept formation that began with visual representations tended to make students focus their attention on visible characteristics, so that the process of generalizing to essential attributes had not yet developed optimally. At this layer, teacher support helped students reflect on the representations they had constructed, thereby laying the foundation for fostering further understanding.

3. Image Having

Students began to use this definition to identify the properties of a parallelogram, indicating that they had reached the image having layer. This development was marked by students' ability to use visual representations as a reference for identifying the properties of geometric figures. This process began with recalling the established definition, after which students used visual representations to identify pairs of sides that were equal in length and parallel.

The argumentation began with data in the form of the definition of a parallelogram, which included the main characteristics of the figure, and then continued with the identification of pairs of sides that were equal in length and parallel as the basis for constructing the argument. The relationship between the data and the conclusion was supported by an warrant explicitly stated fundamental principle, namely the identification of equal-length sides, as well as by an implicit fundamental principle involving the recalled definition of a parallelogram. Based on this line of argumentation, the student asserted that opposite angles were equal and described the properties of a parallelogram's diagonals. In the diagram, solid lines indicated explicitly stated relationships in the argument, while dashed lines indicated implicit relationships. Blue represented the student's contributions, while green indicated the teacher's contributions to the argumentation process.

This was evident when the students read the problem: "Prove that in parallelogram ABCD, AB=DC and AD=BC." After that, the students identified the pairs of sides to be compared by stating, "Oh, I see, AB and DC. Where's the ruler?" They then took measurements and concluded, "3.5." The students then continued with, "And AD is the same length as BC, right? That must be true, right?" and reinforced their observations by stating, "So that means AB is equal to DC. Let's measure it again… see, they're the same. AD is parallel to BC too… so AD is equal to BC." This sequence of argumentation illustrated that students were no longer focusing on listing the properties of a parallelogram but began to use the representations they already possessed to identify relationships between sides through measurement and observation of the diagram.

The characteristics of the students' argumentation indicated that their understanding had progressed to the image having layer. At this layer, students had a mental image of the concept of a parallelogram and were able to use that image to identify relationships between sides in different situations. The representations previously constructed at the image making layer were now used as a reference to recognize corresponding pairs of sides and verify the relationships between them. Although their proofs still relied on measurement results and visual representations, the students had demonstrated that their understanding was no longer limited to figure recognition but began to serve as a foundation for identifying relevant properties. Thus, the students' mental representations began to function as tools to foster the reasoning process regarding the concept of a parallelogram.

At the image having layer, mental representations that had already been formed could be reused without reconstructing previous experiences. Furthermore, Parameswaran (2010) argued that the use of visual representations and gestures to identify relationships between mathematical objects constitutes a first level of abstraction that helps students communicate and foster their understanding of a concept.

4. Property Noticing

Students' attention then shifted from visual representations to the properties that defined the concept of a parallelogram. At this layer, students not only used the representations they already possessed but also began to reexamine relevant attributes to determine the characteristics of a parallelogram.

The argument began with data consisting of the information that a parallelogram had two diagonals of unequal length, supported by a visual representation of the observed figure and the statement that opposite sides are of equal length and opposite angles are equal in measure. Based on this data, the student stated that ACBD was a parallelogram and then identified the properties of the figure $AC \neq BD$ through the relationships $AD=BC$, $AB=DC$, $\angle A=\angle C$, and $\angle B=\angle D$. The relationship between the data and the claim that the figure was a parallelogram was supported by warrant: measuring the diagonals of the figure, visualizing and recalling the properties of a parallelogram, and recognizing that the figure shares three properties of a parallelogram.

There was a process of selecting and clarifying the properties of a parallelogram that were considered appropriate. This was evident when students corrected the statement "the two diagonals are not equal in length, not that two pairs of sides are not equal in length" as a way of clarifying the property being discussed. Furthermore, students re-identified the properties of a parallelogram through the statements "this… has two pairs of equal angles" and "one pair of acute angles and one pair of obtuse angles." When they were asked to list the properties of a parallelogram based on the obtained results, the students again emphasized "having two intersecting diagonals of equal length" and confirmed this identification with the response "yes." This sequence of argumentation indicated that the students' focus began to shift from merely using geometric representations to identifying attributes considered important in explaining the characteristics of a parallelogram. During this process, the students also began to correct and confirm the ideas that emerged, so that the properties used were the result of collective negotiation within the group.

This development in argumentation demonstrated the characteristics of the property noticing layer, in which students began to realize that a mathematical concept was built upon certain properties and used those properties as a basis for recognizing mathematical objects. However, students' attention remained focused on the attributes visible in the observed representations, so they were not yet fully able to distinguish between essential properties and those that were merely consequences of the definition. In other words, students showed progress from using representations to paying attention to relationships among properties, but the understanding they developed was still influenced by their concept images.

Other findings also indicated that concept image and concept definition mutually influenced how students identified and classified mathematical objects (Tirosh & Tsamir, 2022; Vinner, 2002). At this layer, teacher support helped students focus their attention on relevant properties so that their understanding could be fostered toward using those properties as a basis for constructing reasoning and justifications at the next level.

5. Formalising

Students' understanding continued to be fostered at the formalising layer when they began to use the definition and properties of a parallelogram as a basis for justifying the results of their classification of figures. At this layer, students no longer merely recognized or identified the properties of figures, but began to use those properties as a reference to determine whether a figure meets the definition of a parallelogram. The data that students used consisted of the properties of a parallelogram, which were opposite sides of equal length and parallel to each other, two pairs of opposite angles of equal measure, and two diagonals of unequal length, accompanied by a visual representation of the observed figure. Students concluded that the figure was not a parallelogram. The relationship between the data and the conclusion was supported by a fundamental principle, namely comparing the figure's diagonals with the properties of a parallelogram's diagonals and recognizing that the figure did not exhibit any of the three defining properties of a parallelogram.

Students began using the definition of a parallelogram as a reference for evaluating whether a figure's properties matched the definition. This was evident when a student stated, "So, yes, because it has two pairs of equal-length, parallel sides and two pairs of equal opposite angles," and then added, "its two diagonals intersect and are equal in length, and it has no line of symmetry." This line of reasoning demonstrated that students were no longer merely listing the properties of a parallelogram but were beginning to compare the properties of the observed figure with the agreed-upon definition. When one of the properties did not match, students concluded that the figure was not a parallelogram. This process showed that students were using the definition as the basis for evaluating the properties of figures, as visualized through a line of argumentation.

Using the definition as a basis for evaluating whether properties match demonstrated that students no longer viewed the properties of a parallelogram as a list of isolated characteristics. Students began to understand that a figure can be classified by comparing its properties to the formal definition of that figure. In this process, students linked several properties simultaneously to support their classification decisions, so that determining the type of figure no longer depended on a single specific feature, but rather on the interrelationships among the properties that form a definition.

The data used included the properties of the diagonals of a rectangle (AC=BD). the existence of a figure that had similarities to a parallelogram but possessed additional properties, and the properties of the diagonals of a parallelogram (AC≠BD). The warrant linking the data to the conclusion was demonstrated through an activity comparing the diagonals of the two figures. Based on this data and the warrant, students concluded that a rectangle is not a parallelogram.

Students began applying the definition of a parallelogram to evaluate the characteristics of other plane figures. This was evident when a student stated, "No, because a parallelogram has diagonals that intersect at equal lengths, not at right angles." This statement indicated that students were no longer making decisions solely based on the visual appearance of the figure but were using one of the properties of a parallelogram as a reason to reject the classification of that figure. By comparing the characteristics of the diagonals in the observed figure with the definition of a parallelogram, students began to construct justifications based on the correspondence of properties, rather than merely on the results of their observations. This activity demonstrated that the definition served as a reference for evaluating a concept, thereby making the arguments they constructed more systematic.

The students' argumentation structure consisted of a list of quadrilateral types, which were squares, rectangles, parallelograms, rhombuses, kites, and trapezoids, along with the classifications the students had created. The principle used was the simplest common feature shared by these figures, namely that they are all quadrilaterals with four sides, and that squares and rectangles both have four right angles. This data and principle served as the basis for the students in organizing the quadrilaterals into groups.

The students' activities expanded their understanding of the definition of a parallelogram by comparing the relationships among various quadrilaterals. This was evident when a student asked, "A trapezoid is similar to a rectangle, right?" indicating that they were beginning to identify common characteristics among quadrilaterals. Next, students revised their observations by stating, "But this one isn't similar to that one, because only this one resembles a parallelogram," thus demonstrating a process of reevaluating the relationships between figures based on their properties. This line of argumentation demonstrated that students were no longer using definitions merely to identify a single figure but were beginning to use them to compare the degrees of similarity among several quadrilaterals. Thus, students began to organize the relationships between figures based on their characteristics, allowing their understanding to be fostered from a process of classification toward recognizing the interconnections among geometric concepts.

The formalising layer is characterized by students' ability to not only recognize or identify the properties of a parallelogram but also to begin using those properties as a basis for justifying the decisions they make. The definition previously constructed through the process of identifying and connecting various attributes was now used as a reference for evaluating the suitability of a figure's characteristics. This shift was evident when students used the word "because" to connect the properties of a figure to the resulting conclusion. Students' arguments were no longer merely a list of properties but demonstrated logical relationships between the definition, properties, and the classification decisions they made. Thus, the reasoning presented by students was still influenced by their concept images, so some justifications did not fully align with the formal definition of a parallelogram.

During the process of formulating their justifications, students repeatedly recalled and applied the properties of parallelograms discussed in the previous layer before drawing conclusions. This process demonstrated folding back, which occurred when students returned to their existing understanding to reconstruct it before moving on to the next layer. Folding back is a natural part of the development of understanding that allows students to revisit previous ideas when they encounter situations requiring more complex reasoning.

Students began to use mathematical definitions and properties as a basis for constructing arguments and defending their conclusions. These findings were also supported by Tirosh and Tsamir (2022), who found that the development of geometric understanding is marked by students' ability to use essential attributes as a basis for justification, even though their pre-existing concept image still influenced how they interpreted a concept. Furthermore, Birgin & Özkan (2024) explained that the ability to connect definitions with the properties of geometric figures serves as a crucial foundation for fostering a hierarchical understanding of quadrilateral classification. These findings indicated that during the formalizing layer, students began to shift from relying on visual representations to using mathematical definitions and properties as the basis for constructing justifications regarding the relationships among quadrilaterals.

6. Observing

Students' understanding continued to be fostered at the observing layer, as they began to revisit the reasons and relationships among properties that had been established in the previous layer. At this layer, students not only used definitions as a basis for justification but also began to observe the relationships among quadrilaterals more comprehensively.

Students' argumentation based on the data used consists of the information that side AB is opposite to DC, side AD is opposite to BC, and that the pairs of opposite sides have equal lengths, namely AB=DC and AD=BC, supported by the observed geometric representation. The resulting claim stated that AB=DC and AD=BC. The relationship between the data and the claim was supported by two principles: measuring the lengths of the sides of the figure and translating the figure to prove the congruence of the pairs of opposite sides.

At this layer, students began comparing several quadrilaterals to foster connections between concepts. This was evident when students stated that "quadrilaterals are divided into squares and rectangles," then explained that "the simplest, most basic figures still have sides of equal length" and "their angles are also all 90 degrees." These statements indicated that students had attempted to group figures based on characteristics they considered to be the same. However, when explaining the relationships among quadrilaterals, the students still stated, "because a square is similar to a rhombus." This argumentation indicated that the relationships established were still based on observed similarities in figure, rather than on the fulfillment of all the properties underlying the hierarchical relationships among quadrilaterals. Thus, the students began to connect several geometric concepts within a single framework of understanding, but these relationships had not yet been fully structured based on the formal definitions of each geometric figure.

The development of this line of argument showed characteristics of the observing layer, in which students began to reexamine and reflect on the relationships between concepts established in the previous layer, thereby allowing them to see the connections among mathematical objects more broadly. However, this reflective process was still influenced by the students' concept images, so the relationships they established were not yet fully based on the essential properties of each figure. In other words, students were able to recognize the connections between quadrilaterals but had not yet fully used formal definitions as the basis for constructing these relationships.

These findings were also supported by Alcock and Simpson (2017), who stated that the identification of a concept is often influenced by its similarity to familiar examples (the prototype effect), so students do not always use formal definitions as the basis for classification. Furthermore, Tandililing et al. (2025) found that students still struggle to understand the hierarchical relationships among quadrilaterals because they tend to rely on visual characteristics rather than on relationships established through formal definitions and defining properties.

7. Structuring

The characteristics of the structuring layer had not yet emerged in the students' developing understanding. This was evident when a student stated, "The length of AB is equal to DC," and then confirmed, "In the parallelogram… AB is the same length as DC." These statements indicate that students used information obtained from visual representations as the basis for drawing conclusions. Although students had been able to identify pairs of opposite sides, the reasons they gave did not yet show a connection between the definition, properties, or mathematical relationships that could be used to construct a deductive proof. Students' arguments still relied on empirical evidence obtained through observation of the figure, rather than on a logical structure of reasoning.

The structuring level is characterized by students' ability to organize various mathematical ideas into an interconnected structure, enabling them to link definitions, properties, and theorems in constructing proofs. This characteristic was not yet evident in this study because students were not yet able to provide logical reasoning that linked the relationships among the properties of parallelograms to support their conclusions. This indicated that although students had progressed to the point of being able to provide justifications at the formalising layer and reflect on the relationships among concepts at the observing layer, the reasoning structure they had constructed had not yet developed into organized deductive proofs.

These findings were consistent with those of Budiarto and Artiono (2019), who stated that students' geometric proofs are generally still dominated by visual observation and concrete examples before being fostered into deductive proofs based on definitions and relationships among geometric properties. Thus, the trajectory of students' understanding in this study progressed to the observing layer, while characteristics of the structuring layer did not emerge during the observed collective reasoning process.

B. Trajectory of Student Understanding

The results of the analysis revealed a pattern of development in students' understanding of the concept of parallelism during the process of defining and classifying quadrilaterals. Figure 9 presents the trajectory of students' understanding based on Pirie and Kieren's theory. Each trajectory represents important arguments that emerged during the discussion process, while the direction of the arrows indicates movement between layers of understanding. Arrows pointing back to the previous layer indicate a folding-back process, in which students reactivate prior knowledge to foster a deeper understanding. The trajectory demonstrates that the development of understanding progresses from primitive knowing to structuring, while the characteristics of inventising have not yet been identified.

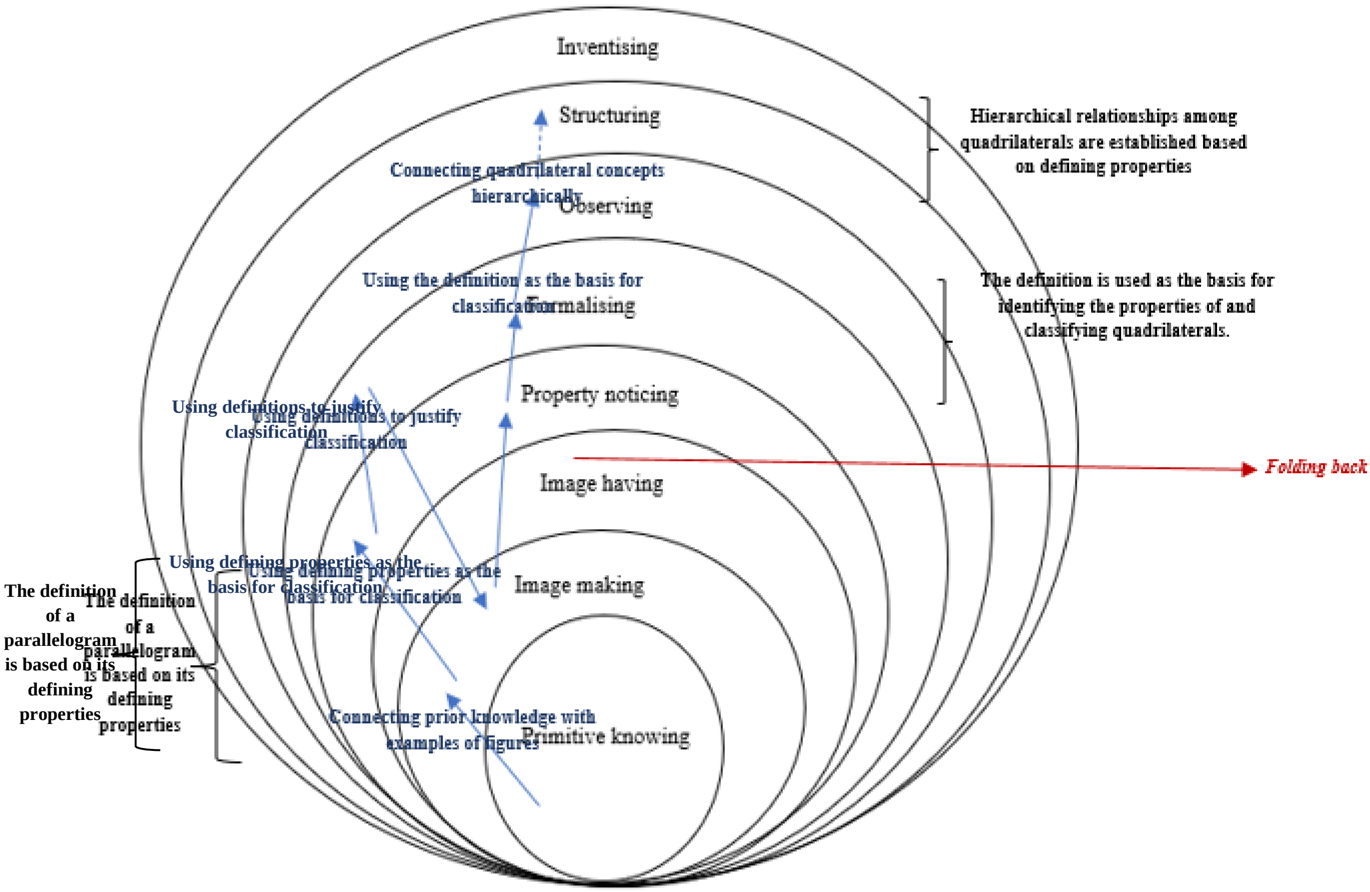

Figure 2. Layers of Understanding

The trajectory shows that the development of understanding begins when students reactivate their prior knowledge of the properties of parallelograms as a foundation for formulating a definition. This prior knowledge is then fostered through the process of observing various examples of quadrilaterals, discussing their properties, and collaboratively formulating a definition. The resulting definition is then used to identify the properties of parallelograms, evaluate attributes that align with or deviate from the concept, and establish a foundation for the classification process. This development indicates that understanding does not increase solely through the presentation of new information, but rather through the reorganization of existing knowledge, leading to the formation of increasingly structured conceptual representations. This finding aligned with research by Irvine (2023), which demonstrated that the development of understanding within the framework proposed by Pirie and Kieren occurs through the process of reconnecting various mathematical representations and ideas to form an increasingly integrated understanding.

As the discussion progressed, students began to use the definition and properties of a parallelogram as a basis for classifying various quadrilaterals. Students' attention was no longer focused solely on visual form but began to be directed toward whether each figure met the specified properties. The definition they had formulated served as a reference when students compared pairs of sides, diagonals, and opposite angles to determine whether a figure met the properties of a parallelogram. In the next layer, students began to establish relationships between squares, rectangles, rhombuses, and parallelograms based on their shared properties. However, the relationships established were not yet fully hierarchical, as some of the reasoning was still influenced by visual similarities. This indicated that students' understanding had progressed toward using definitions as the basis for reasoning, but the conceptual structure that had formed was not yet fully organized (George & Voutsina, 2024).

Overall, this demonstrates that the development of understanding is fostered gradually through a process of constructing definitions, using those definitions to identify properties, and finally connecting various quadrilateral concepts into a unified understanding. This trajectory does not represent a linear progression but rather a dynamic development consistent with the characteristics of Pirie and Kieren's theory. These results reinforced the findings of Chuene et al. (2023) that the growth of mathematical understanding develops through flexible transitions between layers, with each layer serving as the foundation for the formation of understanding in the next layer, rather than as a standalone layer.

C. Folding Back in the Foster of Students' Understanding

Folding back occurs during the transition from the formalising layer to the image-making layer. During the argumentation process, students repeatedly returned to previous layers of understanding before continuing their reasoning to a higher layer. This movement was evident when students were asked to provide reasons for their classification of relationships among quadrilaterals. In such situations, students re-examined the figures, such as by checking the proportionality of side lengths, the parallelism of sides, or the size of angles through observation or measurement, before constructing a more complete argument. These activities helped students reconfirm the necessary information before continuing their reasoning at a higher level of understanding. This indicated that the knowledge built at the initial layer was still used as a foundation for fostering understanding at the next layer.

The folding back process was also evident when students had difficulty determining the characteristics of a figure. Before making a decision, students revisited basic attributes, such as pairs of parallel sides, the lengths of opposite sides, and diagonal properties, and then reused that information to evaluate whether a figure met the definition of a parallelogram. This process of returning to a previous layer did not indicate a regression in understanding but rather served as a mechanism that helped students reorganize their knowledge so they could construct stronger arguments. These findings revealed that the development of understanding did not occur through the gradual addition of information but rather through the reorganization of the relationships among concepts that students already possessed.

Although the folding back process fostered the development of students' understanding, the learning trajectories in this study did not yet demonstrate characteristics of structuring. Students were able to use the definition and properties of parallelograms to identify, classify, and connect several quadrilaterals. However, their arguments still relied on visual observations and measurements, thus failing to form a comprehensive system of deductive reasoning that connects the definition, properties, and relationships among concepts. This indicated that students had achieved the ability to use concepts formally in the classification process but were not yet able to organize various geometric concepts into an interconnected knowledge structure.

The results of this study indicated that folding back served as a mechanism that enabled students to reorganize their prior knowledge before fostering a more complex understanding. These findings were consistent with those of Chuene et al. (2023) and Rivella et al. (2021), who explained that the development of mathematical understanding does not occur through the linear accumulation of facts but rather through the process of reconnecting various ideas to form an increasingly well-developed conceptual network. Similarly, Johnson et al. (2021) argued that understanding develops when students are able to transfer and reorganize relationships among concepts into new situations, rather than merely recalling learned procedures. The trajectory obtained in this study illustrated this process, in which students repeatedly observed and measured the attributes of figures to verify their identifications before using that information to construct classifications and hierarchical relationships among quadrilaterals.

These findings further expand upon the results of the study by Widyastuti et al. (2023), which demonstrated that the growth of mathematical understanding develops through a process of reflection and revision of ideas that emerge during learning interactions. In this study, the revision process was evident through collective reasoning as students revisited the definition, corrected their application of geometric properties, and refined the reasoning used in the classification process. Thus, folding back not only demonstrated the characteristics of understanding development according to Pirie and Kieren's theory but also illustrated how argumentation served as a means for students to foster increasingly integrated conceptual connections in understanding the definition and classification of quadrilaterals.

IV. CONCLUSION

This study shows that students' understanding of the definition and classification of quadrilaterals is dynamically fostered during the process of collective argumentation. Students' understanding developed through the process of connecting, revising, and reinforcing their existing concepts as they discussed various ideas that emerged within their groups. Based on Pirie and Kieren's theory, this development did not occur linearly but rather through a process of folding back, in which students revisited their prior knowledge to establish more meaningful connections between the definitions, properties, and classifications of quadrilaterals.

The development of understanding is not always reflected in the resulting argumentative structures. Students may construct similar argumentative patterns but differ in their levels of understanding due to variations in how they connect geometric concepts and reorganize their existing knowledge. These findings suggest that the quality of understanding is determined more by the depth of the conceptual connections that are successfully established than by the similarity of the argumentative forms.

These findings indicate that collective argumentation serves as a vehicle for fostering mathematical understanding, rather than merely a means of presenting answers. Through the process of questioning one another, reasoning, evaluating, and revising ideas, students have the opportunity to negotiate the meaning of concepts and strengthen the relationships between the definitions, properties, and classifications of quadrilaterals. Thus, learning that provides space for argumentation allows students to foster a more structured and meaningful understanding.

In general, this study shows that combining an analysis of the development of understanding with Toulmin's argumentation model provides a more comprehensive picture of the mathematics learning process. The analysis of the development of understanding helps explain the dynamics of transitions between layers of understanding, while Toulmin's model helps identify how the argumentation process fosters that development. These results indicate that the analysis of mathematics learning needs to consider not only the quality of visible argumentation but also the conceptual changes that occur as students construct their understanding.

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