

Doubly Connected 2-Domination in the Join of Two Graphs

Mylyn L. Doren¹, Abraham P. Racca²

^{1,2}Professional Schools University of Mindanao Davao City, Philippines

²Mathematics and Physics Department Adventist University of the Philippines Silang, Cavite, Philippines

ABSTRACT: Given a nontrivial connected graph, a set of vertices of a graph is a doubly connected 2-dominating set if the induced subgraphs of the set and its complement are connected, and every vertex in its complement is adjacent to at least two vertices in the set. The doubly connected 2-domination number of a nontrivial connected graph is the cardinality of a minimum doubly connected 2-dominating set. In this paper, we characterize the doubly connected 2-dominating sets in the join of two graphs and derive the corresponding doubly connected 2-domination number.

KEYWORDS: doubly connected 2-domination, connected 2-domination, 2-domination, domination, join.

1 INTRODUCTION

Doubly connected 2-domination in a connected graph is a combination of the concepts of connected 2-domination and outer-connected domination. Doren and Leonida [2] introduced and explored the concept of doubly connected 2-domination. They showed properties and bounds of the doubly connected 2-domination number. Moreover, they exhibited relationship of this parameter with other domination parameters. In this paper, we characterize the doubly connected 2-dominating set in the join of two graphs and derive the corresponding doubly connected 2-domination number.

2 TERMINOLOGY AND NOTATION

Let $G = (V(G), E(G))$ be a graph, where $V(G)$ is the *vertex-set* of G and $E(G)$ is the *edge-set* of G . We call v a *vertex* of G if $v \in V(G)$ and uv is an *edge* of G if $uv \in E(G)$. Two vertices u and v of G are said to be *adjacent* or *neighbors* if uv is an *edge* of G . The *open neighborhood* of v in G , denoted by $N_G(v)$, is a set of neighbors of a vertex v in G . The set $N_G[v] = \{v\} \cup N_G(v)$ is called the *closed neighborhood* of v in G .

A *path* in a graph G is a sequence of distinct vertices v_0, v_1, \dots, v_n , where there is an edge between each consecutive pair of vertices. The length of a path is the number of edges within it. A $u-v$ path is a path whose initial vertex is u and terminal vertex is v . A graph G is called a *connected* graph if every two vertices in G are joined by a path; otherwise, it is called a *disconnected* graph.

The *degree* of a vertex v of G , denoted by $\deg_G(v)$, is the number of neighbors of v in G , that is, $\deg_G(v) = |N_G(v)|$. The *distance* between vertices u and v , denoted by $d_G(u, v)$, is the length of the shortest $u-v$ path in G .

A subset C of $V(G)$ is connected if the subgraph $\langle C \rangle$ induced by C is connected.

The *join* of two graphs G and H , denoted by $G+H$, is the graph with vertex-set $V(G+H) = V(G) \cup V(H)$ and edge-set $E(G+H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

A subset S of $V(G)$ is a *dominating set* in G if for every $v \in V(G) \setminus S$, there exists $u \in S$ such that $uv \in E(G)$. The *domination number* of G , denoted by $\gamma(G)$, is the cardinality of a minimum dominating set in G .

Given a connected graph G , a subset C of $V(G)$ is called a *connected dominating set* in G if C is a connected set and a dominating set. The *connected domination number* of G , denoted by $\gamma_c(G)$, is the cardinality of a minimum connected dominating set in G . A subset C of $V(G)$ is called an *outer-connected dominating set* in G if C is a dominating set and the subgraph $\langle V(G) \setminus C \rangle$ is connected. The *outer-connected domination number* of G , denoted by $\gamma_{ec}(G)$, is the cardinality of a minimum outer-connected dominating set in

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A subset D of $V(G)$ is a *2-dominating set* in G if every $v \in V(G) \setminus D$, $|D \cap N_G(v)| \geq 2$, that is, v is adjacent to at least two vertices in D . The *2-domination number* of G , denoted by $\gamma_2(G)$, is the cardinality of a minimum 2-dominating set in G . A subset C of $V(G)$ is a *connected 2-dominating set* in G if C is connected and a 2-dominating set. The *connected 2-domination number* of G , denoted by $\gamma_{2c}(G)$, is the cardinality of a minimum connected 2-dominating set in G .

Given a nontrivial connected graph G , a subset C of $V(G)$ is called a *doubly connected 2-dominating set* in G if $\langle C \rangle$ and $\langle V(G) \setminus C \rangle$ are connected and C is a 2-dominating set. The *doubly connected 2-domination number* of G , denoted by $\gamma_{2cc}(G)$, is the cardinality of a minimum doubly connected 2-dominating set in G .

Corollary 2.1 Let G be a nontrivial connected graph.

- (i) If C_0 is a minimum doubly connected 2-dominating set in G , then $\gamma_{2cc}(G) = |C_0|$.
- (ii) If C is a doubly connected 2-dominating set in G , then $\gamma_{2cc}(G) \leq |C|$.

Theorem 2.2 [2] Let G be a connected graph of order $n \geq 2$. Then $2 \leq \gamma_{2cc}(G) \leq n$.

3 DOUBLY CONNECTED 2-DOMINATION IN THE JOIN OF TWO GRAPHS

In this section we characterize the doubly connected 2-dominating set in the join of two graphs and derive the doubly connected 2-domination number.

Theorem 3.1 Let G and H be graphs each of order at least 2. Then a subset C of $V(G+H)$ is a doubly connected 2-dominating set in $G+H$ if and only if it one of the following holds:

- (i) $C = V(G)$ or $C = V(H)$, where both G and H are connected graphs.
- (ii) $C = V(G)$ and C is a connected 2-dominating set in G .
- (iii) $C = V(H)$ and C is a connected 2-dominating set in H .
- (iv) $|C \cap V(G)| = |C \cap V(H)| = 1$, and $C \cap V(G)$ is a dominating set in G and $C \cap V(H)$ is a dominating set in H .
- (v) $|C \cap V(G)| \geq 2$, and $|C \cap V(H)| = 1$ and $C \cap V(G)$ is a dominating set in G .
- (vi) $|C \cap V(H)| \geq 2$, and $|C \cap V(G)| = 1$ and $C \cap V(H)$ is a dominating set in H .
- (vii) $|C \cap V(G)| \geq 2$ and $|C \cap V(H)| \geq 2$.
- (viii) $C \cap V(G) = V(G)$ and $\langle V(H) \setminus (C \cap V(H)) \rangle$ is a connected subgraph of H .
- (ix) $C \cap V(H) = V(H)$ and $\langle V(G) \setminus (C \cap V(G)) \rangle$ is a connected subgraph of G .

Proof: Suppose that C is a doubly connected 2-dominating set in $G+H$. Then C is a 2-dominating set in $G+H$ and both $\langle C \rangle$ and $\langle V(G+H) \setminus C \rangle$ are connected subgraphs of $G+H$. Consider the following cases:

Case 1. $C \cap V(G) = \emptyset$ or $C \cap V(H) = \emptyset$ but not both.

Then either $C \subseteq V(G)$ or $C \subseteq V(H)$. Consider the following subcases: Subcase 1.1. $C = V(G)$ or $C = V(H)$.

Suppose that $C = V(G)$. Then $V(G+H) \setminus C = V(H)$. By hypothesis, C is a 2-dominating set in $G+H$ and both $\langle C \rangle$ and $\langle V(G+H) \setminus C \rangle$ are connected subgraphs of $G+H$. But $C = V(G)$ and $V(G+H) \setminus C = V(H)$. This implies that both $\langle V(G) \rangle = G$ and $\langle V(H) \rangle = H$ are connected subgraphs of $G+H$. Hence, G and H are connected graphs. Similarly, if $C = V(H)$, then H and G are connected graphs. This proves (i).

Subcase 1.2. $C \neq V(G)$.

Then C is a proper subset of $V(G)$. But C is a 2-dominating set in $G+H$ and both $\langle C \rangle$ and $\langle V(G+H) \setminus C \rangle$ are connected subgraphs of $G+H$. Thus, C is a 2-dominating set in G and $\langle C \rangle$ is a connected subgraph of G . Hence, C is a connected 2-dominating set in G . This proves (ii). Similarly, if $C \neq V(H)$, then C is a connected 2-dominating set in H . This proves (iii).

Case 2. $C \cap V(G) \neq \emptyset$ and $C \cap V(H) \neq \emptyset$.

Then $|C \cap V(G)| \geq 1$ and $|C \cap V(H)| \geq 1$. Consider the following subcases: Subcase 2.1. $|C \cap V(G)| = 1$ and $|C \cap V(H)| = 1$.

Let $C \cap V(G) = \{u\}$ and $C \cap V(H) = \{v\}$. Then $C = \{u, v\}$ and $uv \in E(G+H)$. We show that $C \cap V(G) = \{u\}$ and $C \cap V(H) = \{v\}$ are dominating sets in G and H , respectively. Let $x \in V(G+H) \setminus C$. Then either $x \in V(G+H) \setminus (C \cap V(G))$ or $x \in V(G+H) \setminus (C \cap V(H))$. If $x \in V(G+H) \setminus (C \cap V(G))$, then $xv \in E(G+H)$. Since C is a 2-dominating set in $G+H$, x is dominated by at least 2 vertices in $G+H$. This implies that there exists $y \in C \cap V(G)$ such that $xy \in E(G)$. Hence, $C \cap V(G) = \{u\}$ is a dominating set in G . Similarly, if $x \in V(G+H) \setminus (C \cap V(H))$, then $C \cap V(H) = \{v\}$ is a dominating set in H . This proves (iv).

Subcase 2.2. $|C \cap V(G)| = 1$ and $|C \cap V(H)| > 1$.

Then $|C \cap V(G)| = 1$ and $|C \cap V(H)| \geq 2$. Let $C \cap V(G) = \{u\}$ and let $w \in V(H) \setminus (C \cap V(H))$. Then $C = \{u\} \cup (C \cap V(H))$. Suppose that $C \cap V(H)$ is not a dominating set in H . Then w is not dominated by any vertex in $C \cap V(H)$. Note that $uw \in E(G+H)$. This implies that w is dominated by only one vertex in C . This contradicts the fact that C is a 2-dominating set in $G+H$. Therefore, $C \cap V(H)$ is a dominating set in H . This proves (v). Similarly, if $|C \cap V(G)| >$

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1 and $|C \cap V(H)| = 1$, then $C \cap V(G)$ is a dominating set in G . This proves (vi).

Subcase 2.3. $|C \cap V(G)| > 1$ and $|C \cap V(H)| > 1$.

Then $|C \cap V(G)| \geq 2$ and $|C \cap V(H)| \geq 2$. This proves (vii). Subcase 2.4. $|C \cap V(G)| > 1$ and $|C \cap V(H)| = |V(H)|$.

Then $|C \cap V(G)| \geq 2$ and $|C \cap V(H)| = |V(H)|$. This implies that $V(G+H) \setminus C = V(G) \setminus (C \cap V(G))$ and so $\langle V(G+H) \setminus C \rangle = \langle V(G) \setminus (C \cap V(G)) \rangle$.

Since, C is a doubly connected 2-dominating set in $G+H$, it follows $\langle V(G+H) \setminus C \rangle$ is a connected subgraph of $G+H$. Hence, $\langle V(G) \setminus (C \cap V(G)) \rangle$ is a connected subgraph of G . This proves (viii). Similarly, if $|C \cap V(H)| > 1$ and $|C \cap V(G)| = |V(G)|$, then $\langle V(H) \setminus (C \cap V(H)) \rangle$ is a connected subgraph of H .

This proves (ix).

For the converse, firstly, assume that (i) holds. If $C = V(G)$, then $V(G+H) \setminus C = V(H)$. Thus, $\langle C \rangle = G$ and $\langle V(G+H) \setminus C \rangle = H$. Since G and H are connected graphs, it follows that $\langle C \rangle$ and $V(G+H) \setminus C$ are connected subgraphs of $G+H$. Let $x \in V(G+H) \setminus C$. Then $x \in V(H)$ since $V(G+H) \setminus C = V(H)$. Since $|V(G)| \geq 2$, x is adjacent to at least two vertices in $C = V(G)$. This implies that C is a 2-dominating set in $G+H$. Hence, C is a doubly connected 2-dominating set in $G+H$. Similarly, if $C = V(H)$, then C is a doubly connected 2-dominating set in $G+H$.

Secondly, assume that (ii) holds. Then C is a 2-dominating set in G

and $\langle C \rangle$ is a connected subgraph of $G+H$. We need only to show that

$\langle V(G+H) \setminus C \rangle$ is a connected subgraph of $G+H$. Note that $V(G) \setminus C \neq \emptyset$. Thus, $V(G+H) \setminus C = (V(G) \setminus C) \cup V(H)$, which implies that $\langle V(G+H) \setminus C \rangle =$

$\langle V(G) \setminus C \rangle + H$. Hence, $\langle V(G+H) \setminus C \rangle$ is a connected subgraph of $G+H$. This implies that C is a doubly connected 2-dominating set in $G+H$. Similarly, if

(iii) holds, that is, $C = V(H)$ and C is a connected 2-dominating set in H ,

then C is a doubly connected 2-dominating set in $G+H$.

Thirdly, assume that (iv) holds. Let $C = \{u, v\}$, where $C \cap V(G) = \{u\}$ and $C \cap V(H) = \{v\}$. Then $uv \in E(G+H)$, which implies that $\langle C \rangle$ is a connected subgraph of $G+H$. Since $|V(G)| \geq 2$ and $|V(H)| \geq 2$, it follows that $V(G+H) \setminus C$ is a nonempty subset of $V(G+H)$. But $\langle V(G+H) \setminus C \rangle$ is a connected subgraph of $G+H$. Let $x \in V(G+H) \setminus C$. Then either $x \in V(G) \setminus (C \cap V(G))$ or $x \in V(H) \setminus (C \cap V(H))$. If $x \in V(G) \setminus C$, then since $C \cap V(H)$ is a dominating set in G . Thus, $vx \in E(G+H)$. This implies that x is dominated by 2 vertices in C . Hence, C is a 2-dominating set in $G+H$. Similarly, if $x \in V(H) \setminus (C \cap V(H))$, then C is a 2-dominating set in $G+H$. Therefore, C is a doubly connected 2-dominating set in $G+H$.

Fourthly, assume that (v) holds. Let us define $C = (C \cap V(G)) \cup \{w\}$, where $C \cap V(G)$ is a minimum dominating set in G and $C \cap V(H) = \{w\}$. Then $\langle C \rangle = \langle \{w\} \rangle + \langle V(G) \setminus (C \cap V(G)) \rangle$, which implies that $\langle C \rangle$ is a connected subgraph of $G+H$. Moreover, $V(G+H) \setminus C = (V(G) \setminus (C \cap V(G))) \cup (V(H) \setminus (C \cap V(H)))$. Thus, $V(G+H) \setminus C$ is a nonempty subset of $V(G+H)$ and $\langle V(G+H) \setminus C \rangle$ is a connected subgraph of $G+H$. Let $x \in V(G+H) \setminus C$. Then either $x \in V(G) \setminus (C \cap V(G))$ or $x \in V(H) \setminus (C \cap V(H))$. If $x \in V(G) \setminus (C \cap V(G))$, then since $C \cap V(G)$ is a dominating set in G , x is dominated by at least one vertex in C . But by $wx \in E(G+H)$. This implies that x is dominated by at least 2 vertices in C . So that C is a 2-dominating set in $G+H$. Therefore, C is a doubly connected 2-dominating set in $G+H$. If $x \in V(H) \setminus (C \cap V(H))$, then since $|C \cap V(G)| \geq 2$, x is adjacent to at least two vertices in $C \cap V(G) \subseteq C$. Thus, C is a 2-dominating set in $G+H$. Therefore, C is a doubly connected 2-dominating set in $G+H$. Similarly, if (vi) holds, that is, $|C \cap V(H)| \geq 2$, and $|C \cap V(G)| = 1$ and $C \cap V(H)$ is a dominating set in H , then C is a doubly connected 2-dominating set in $G+H$.

Fifthly, assume that (vii) holds. If $|V(G)| = |V(H)| = 2$, then let $C = V(G) \cup V(H) = V(G+H)$. Thus, C is a doubly connected 2-dominating set in $G+H$. So, we assume that $|V(G)| \geq 3$ and $|V(H)| \geq 3$. Let us define $C = \{u, v, w, z\}$ where $u, v \in V(G)$ and $w, z \in V(H)$. Then $C \cap V(G) = \{u, v\}$ and $C \cap V(H) = \{w, z\}$. Since $|V(G+H)| \geq 6$ and $|C| = 4$, it follows that $|V(G+H) \setminus C| \geq 2$. This implies that C and $V(G+H) \setminus C$ are nonempty subsets of $V(G+H)$. Now, both $\langle C \rangle$ and $\langle V(G+H) \setminus C \rangle$ are connected subgraphs of $G+H$. Let $x \in V(G+H) \setminus C$. Then either $x \in V(G) \setminus (C \cap V(G))$ or $x \in V(H) \setminus (C \cap V(H))$. If $x \in V(G) \setminus (C \cap V(G))$, then $xw, xz \in E(G+H)$. Thus, C is a 2-dominating set in $G+H$. Similarly, if $x \in V(H) \setminus (C \cap V(H))$, then C is a 2-dominating set in $G+H$. Therefore, C is a doubly connected 2-dominating set in $G+H$.

Lastly, assume that (viii) holds. This means that $C = V(G) \cup (C \cap V(H))$, and so, C is a nonempty subset of $V(G+H)$. So that $\langle C \rangle$ is a connected subgraph of $G+H$. But, by the assumption, $V(G+H) \setminus C = V(H) \setminus (C \cap V(H))$. Thus, $\langle V(G+H) \setminus C \rangle = \langle V(H) \setminus (C \cap V(H)) \rangle$. Hence, $\langle V(G+H) \setminus C \rangle$ is a connected subgraph of $G+H$. Let $x \in V(G+H) \setminus C$. Then $x \in V(H) \setminus (C \cap V(H))$. Since $|C| = |V(G)| \geq 2$, x is adjacent to at least 2 vertices in C . This implies that C is a 2-dominating set in $G+H$. Therefore, C is a doubly connected 2-dominating set in $G+H$. Similarly, if (ix) holds, that is, $C \cap V(H) = V(H)$ and $\langle V(G) \setminus (C \cap V(G)) \rangle$ is a connected subgraph of G , then C is a doubly connected 2-dominating set in $G+H$.

Corollary 3.2 Let G and H be graphs each of order at least 2. Then

$$2 \leq \gamma_{2cc}(G+H) \leq 4.$$

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Proof: The join $G + H$ is a graph of order at least 4. Thus, by Theorem 2.2, $2 \leq \gamma_{2cc}(G + H)$. Now, let $C = \{a, b, x, y\}$, where $a, b \in C \cap V(G)$ and $x, y \in C \cap V(H)$. Then $|C \cap V(G)| = 2$ and $|C \cap V(H)| = 2$. By Theorem, 3.1, C is a doubly connected 2-dominating set in $G + H$. Thus, by Corollary 2.1(ii), $\gamma_{2cc}(G + H) \leq |C| = 4$. Therefore, $2 \leq \gamma_{2cc}(G + H) \leq 4$.

Corollary 3.3 Let G and H be graphs each of order at least 2. Then

$\gamma_{2cc}(G + H) = 2$ if and only if one of the following holds:

- (i) $\gamma(G) = 1$ and $\gamma(H) = 1$.
- (ii) $\gamma_{2c}(G) = 2$.
- (iii) $\gamma_{2c}(H) = 2$.

Proof: Suppose that $\gamma_{2cc}(G + H) = 2$. Let C be a minimum doubly connected 2-dominating set in $G + H$. By Theorem 3.1, we have the following cases:

Case 1. By Theorem 3.1(i), $|C \cap V(G)| = 1$ and $|C \cap V(H)| = 1$, where

$C \cap V(G)$ is a dominating set in G and $C \cap V(H)$ is a dominating set in H . This implies that $C \cap V(G)$ is a minimum dominating set in G and $C \cap V(H)$ is a minimum dominating set in H . Hence, $\gamma(G) = 1$ and $\gamma(H) = 1$. This proves (i).

Case 2. By Theorem 3.1(ii), $C = V(G)$ and C is a connected 2-dominating set in G . Then C is a minimum doubly connected 2-dominating set in G from the assumption. Hence, by Corollary 2.1(i), $\gamma_{2cc}(G) = 2$. This proves (ii).

Case 3. By Theorem 3.1(iii), $C = V(H)$ and C is a connected 2-dominating set in H . Then C is a minimum doubly connected 2-dominating set in H by assumption. Hence, by Corollary 2.1(i), $\gamma_{2cc}(H) = 2$. This proves (iii).

Conversely, assume that (i) holds, that is, $\gamma(G) = 1$ and $\gamma(H) = 1$. Let $C \cap V(G) = \{u\}$ be a dominating set in G and let $C \cap V(H) = \{v\}$ be a dominating set in H . Set $C = \{u, v\}$. By Theorem 3.1(i), C is a doubly connected 2-dominating set in $G + H$. Thus, by Corollary 2.1(ii), $\gamma_{2cc}(G + H) \leq |C| = 2$. But by Theorem 2.2, $2 \leq \gamma_{2cc}(G + H)$. Therefore, $\gamma_{2cc}(G + H) = 2$. Next, assume that (ii) holds, that is, $\gamma_{2c}(G) = 2$. Let C_0 be a minimum connected 2-dominating set in G . Then, by Theorem 3.1(ii), C_0 is a doubly connected 2-dominating set in $G + H$. By Corollary 2.1(i), $\gamma_{2cc}(G + H) \leq |C_0| = 2$. But by Theorem 2.2, $2 \leq \gamma_{2cc}(G + H)$. Therefore, $\gamma_{2cc}(G + H) = 2$. Similarly, if

(iii) holds, then $\gamma_{2cc}(G + H) = 2$.

Corollary 3.4 Let G and H be graphs each of order at least 2 such that $\gamma_{2cc}(G + H) \neq 2$. Then $\gamma_{2cc}(G + H) = 3$ if and only if one of the following holds:

- (i) $\gamma(G) = 2$.
- (ii) $\gamma(H) = 2$.
- (iii) $\gamma_{2c}(G) = 3$.
- (iv) $\gamma_{2c}(H) = 3$.

Proof: Let G and H be graphs each of order at least 2 with $\gamma_{2cc}(G + H) \neq 2$. Then $\gamma_{2cc}(G + H) \geq 3$, which implies that $\gamma_{2cc}(G) \geq 3$ or $\gamma_{2cc}(H) \geq 3$. Assume that $\gamma_{2cc}(G + H) = 3$. Let C_0 be a minimum doubly connected 2-dominating set in $G + H$. Then $|C_0| = 3$. We have the following cases:

Case 1. $|C \cap V(G)| = 1$ and $C \cap V(H)$ is a dominating set in H . Then $|C \cap V(H)| \geq 2$. Thus, by Corollary 2.2.2(ii), $\gamma(H) \leq |C \cap V(H)| = 2$. But by Theorem 2.3.1, $\gamma(H) \geq 2$. Hence, $\gamma(H) = 2$.

This proves (ii).

Case 2. $|C \cap V(H)| = 1$ and $C \cap V(G)$ is a dominating set in G . By similar arguments in Case 3, we have $\gamma(G) = 2$. This proves (i).

Case 3. By Theorem 3.1(v), $C_0 = V(H)$ and C_0 is a connected 2-dominating set in H . Then C_0 is a minimum doubly connected 2-dominating set in G . Thus, $\gamma_{2c}(G) = |C_0| = 3$. This proves (iv). Similarly, if $C_0 = V(G)$ and C_0 is a connected 2-dominating set in G , then $\gamma_{2c}(H) = 3$. This proves (v).

The converse is clear.

The proof of the next result is similar to Corollary 3.3. Hence, we omit the proof.

Corollary 3.5 Let G and H be graphs each of order at least 2 such that $\gamma_{2cc}(G + H) \neq 2, 3$. Then $\gamma_{2cc}(G + H) = 4$ if and only if one of the following holds:

- (i) $\gamma(G) = 3$.
- (ii) $\gamma(H) = 3$.
- (iii) $\gamma_{2c}(G) = 4$.
- (iv) $\gamma_{2c}(H) = 4$.
- (v) $|C \cap V(G)| = 2$ and $|C \cap V(H)| = 2$.

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