

Independent 2-Domination in the Corona of Graphs

Berlita Y. Disca¹, Abraham P. Racca²

^{1,2}Professional Schools University of Mindanao Davao City, Philippines

¹Mathematics Department College of Natural Sciences and Mathematics Mindanao State University - General Santos Fatima, General Santos City, Philippines

²Mathematics and Physics Department Adventist University of the Philippines Silang, Cavite, Philippines

ABSTRACT: Given a nontrivial graph, a set of vertices of a graph is an independent set if every pair of distinct vertices are not adjacent and it is a 2-dominating set if each vertex in its complement is adjacent to at least two vertices in the set. A set of vertices of a graph is an independent 2-dominating set if it is both an independent set and a 2-dominating set. The independent 2-domination number of a nontrivial graph is the cardinality of a minimum independent 2-dominating set. In this paper, we characterize the independent 2-dominating sets in the vertex and edge corona of graphs and derive their independent 2-domination numbers.

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1 INTRODUCTION

Independent 2-domination in a graph is a combination of the concept of 2-domination and the notion of an independent set of vertices of a graph. In 1962, Berge [1] initiated the concept of independent domination and twelve years later, Cockayne and Hedetnieme [5] introduced the notation for the independent domination number. Goddard and Henning [6], gave an upper bound for the sum of the independent domination numbers of a graph and its complement.

The concept 2-domination was studied by Blidia et. al. [2]. They found properties and bounds on the 2-domination number. In [3], V. Maheswari et.al. derived the 2-dominating sets and 2-domination numbers of some special graphs. Also, Domoloan and Canoy, Jr. [5] obtained the 2-domination numbers of the join and corona of two graphs.

The concept of independent 2-domination was introduced by Leonida and Allosa in [8]. They characterized the independent 2-dominating set of the join of two graphs and obtained its independent 2-domination number. In this paper, we characterize the independent 2-dominating sets in the vertex and edge corona of graphs and derive their independent 2-domination numbers.

2 TERMINOLOGY AND NOTATION

Let $G = (V(G), E(G))$ be a nontrivial graph, where $V(G)$ is the *vertex-set* of G and $E(G)$ is the *edge-set* of G . An element v of $V(G)$ is called a *vertex* and an element uv of $E(G)$ is called an *edge*. Two vertices u and v of G are *adjacent* or *neighbors* if uv is an *edge* of G . The set of neighbors of a vertex v in G , denoted by $N_G(v)$, is called the *open neighborhood* of v in G . The set $N_G[v] = \{v\} \cup N_G(v)$ is called the *closed neighborhood* of v in G .

The *degree* of a vertex v in G , denoted by $\deg_G(v)$, is the number of neighbors of v in G , that is, $\deg_G(v) = |N_G(v)|$. Moreover, v is called an *isolated vertex* if $\deg_G(v) = 0$; v is an *end-vertex* if $\deg_G(v) = 1$; and v is a *support* if v is adjacent to an end-vertex.

A *path* in a graph G is a sequence of distinct vertices v_0, v_1, \dots, v_n , where there is an edge between each consecutive pair of vertices. A graph G is called a *connected* graph if every two vertices in G are joined by a path; otherwise, it is called a *disconnected* graph. Every disconnected graph can be split up into a number of connected subgraphs called *components*.

Let G and H be graphs of order m and n , respectively. The *vertex corona* of G and H , denoted by $G \circ H$, is the graph obtained by taking one copy of G and m copies of H , and then joining the i th vertex of G to every vertex of the i th copy of H . The *edge corona* of G and H , denoted by $G \diamond H$, is the graph obtained by taking one copy of G and $|E(G)|$ copies of H and then joining the two end-vertices of the i th edge of G to every vertex in the i th copy of H .

An $(n, 2)$ -centipede graph $c_{n,2}$ of order $3n$, where $n \geq 1$, is obtained from a path P_n and attaching exactly two pendant edges to each vertex of the path, that is, $c_{n,2} = P_n \circ K_2$, for $n \geq 1$. A *bi-star* graph $B(r, s)$ of order $r + s + 1$, where $r, s \geq 2$, is a graph obtained by joining the centers of two stars $K_{1,r}$ and $K_{1,s}$. A bi-star graph $B(r, r)$ of order $2r + 1$, where $r \geq 2$, is a graph obtained by joining the centers of two copies of the star $K_{1,r}$. A bi-star $B(r, r)$ may be defined by $B(r, r) = K_2 \circ K_r$, for $n \geq 2$.

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A subset S of $V(G)$ is a *dominating set* in G if for every $v \in V(G) \setminus S$, there exists $u \in S$ such that $uv \in E(G)$. The *domination number* of G , denoted by $\gamma(G)$, is the cardinality of a minimum dominating set in G .

A subset S of $V(G)$ is called an *independent set* in G if for every $u, v \in S$, $uv \notin E(G)$, that is, if every pair of vertices in S are not adjacent. A subset S of $V(G)$ is an *independent dominating set* in G if S is an independent set and a dominating set in G . The *independent domination number* of G , denoted by $i(G)$, is the cardinality of a minimum independent dominating set in G .

A subset D of $V(G)$ is a *2-dominating set* in G if every $v \in V(G) \setminus D$, $|D \cap N_G(v)| \geq 2$, that is, v is adjacent to at least two vertices in D . The *2-domination number* of G , denoted by $\gamma_2(G)$, is the cardinality of a minimum 2-dominating set in G .

Definition 2.1 Let G be a nontrivial graph. A subset S of $V(G)$ is an *independent 2-dominating set* in G if S is an independent set in G and a 2-dominating set in G .

Example 2.2 Consider the graph G in Figure 1. Let $S_0 = \{a, c\}$.

- (i) Since $ac \notin E(G)$, it follows that S_0 is an independent set in G .
- (ii) Let $b \in V(G) \setminus S_0$. Then b is adjacent to two vertices a and c in S_0 . Hence, S_0 is a 2-dominating set in G . Therefore, by Definition 2.1, S_0 is an independent 2-dominating set in G .

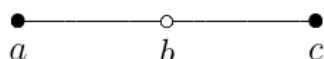


Figure 1: A nontrivial graph G with an independent 2-dominating set.

The following example shows a nontrivial graph with no independent 2-dominating set.

Example 2.3 Consider the graph G in Figure 2.

- (1) The set $S_1 = \{u\}$ is not an independent 2-dominating set in G since it is not a 2-dominating set in G . Similarly, $S_2 = \{v\}$ is not an independent 2-dominating set in G .
- (2) The set $S_3 = \{u, v\}$ is not an independent 2-dominating set in G since it is not an independent set in G because u and v are adjacent vertices. Therefore, G is a nontrivial graph without an independent 2-dominating set.



Figure 2: A nontrivial graph G with no independent 2-dominating set.

In case a nontrivial graph G has an independent 2-dominating set, we shall say " G admits an independent 2-dominating set."

Definition 2.4 Let G be a nontrivial graph. Suppose G admits an independent 2-dominating set. Then the *independent 2-domination number* of G , denoted by $i_2(G)$, is the cardinality of a minimum independent 2-dominating set in G .

The following Corollary follows from Definition 2.4.

Corollary 2.5 Let G be a nontrivial graph.

- (i) If S_0 is a minimum independent 2-dominating set in G , then $i_2(G) = |S_0|$.
- (ii) If S is an independent 2-dominating set in G , then $i_2(G) \leq |S|$.

Theorem 2.6 Let G be a nontrivial graph. If S_0 is a minimum independent 2-dominating set in G , then S_0 contains every isolated vertex of G .

Proof: Let v be an isolated vertex of G . Suppose that $v \notin S_0$. Then $v \in V(G) \setminus S_0$. But v is not adjacent to any other vertex of G . This contradicts the fact that S_0 is a 2-dominating set in G . Therefore, S_0 contains every end-vertex of G .

Corollary 2.7 Let G be a nontrivial graph. Suppose G admits an independent 2-dominating set. Then $i_2(G) \geq |I(G)|$, where $I(G)$ is the set of all isolated vertices of G .

Proof: Let $I(G)$ be the set of all isolated vertices of G and S_0 a minimum independent 2-dominating set in G . Then, by Theorem 2.6, $S_0 \supseteq I(G)$, which implies that $|S_0| \geq |I(G)|$. But $i_2(G) = |S_0|$ by Corollary 2.5(i). Therefore, $i_2(G) \geq |I(G)|$.

The following result follows from Corollary 2.7.

Corollary 2.8 Let K_n be an empty graph of order $n \geq 2$. Then $i_2(K_n) = n$.

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3 Independent 2-Domination in the Vertex Corona of Graphs

In this section we characterize the independent 2-dominating set in the vertex corona and derive the independent 2-domination number.

Theorem 3.1 *Let G be a graph of order $m \geq 2$ and H a graph of order $n \geq 2$. Suppose that H admits an independent 2-dominating set. Then a nonempty subset S of $V(G \circ H)$ is an independent 2-dominating set in $G \circ H$ if and only if*

$$S = \bigcup_{v \in V(G)} S^v,$$

where S^v is an independent 2-dominating set in H^v for all $v \in V(G)$.

Proof: Let S be an independent 2-dominating set in $G \circ H$. We show that $S \cap V(G) = \emptyset$. Suppose that there exists $v \in S \cap V(G)$. Since S is independent, it follows that $S \cap V(G)$ is independent. By the definition of the corona, $vw \in E(G \circ H)$ for all $w \in V(H^v)$. This implies that $S \cap V(H^v) = \emptyset$. This contradicts the fact that S is a 2-dominating set. Thus, $S \cap V(G) = \emptyset$. Accordingly, $S \cap V(H^v) \neq \emptyset$ and $S \cap V(H^v)$ is an independent 2-dominating set in H^v for all $v \in V(G)$. For each $v \in V(G)$, let $S^v = S \cap V(H^v)$.

Therefore,

$$S = \bigcup_{v \in V(G)} S^v,$$

where S^v is an independent 2-dominating set in H^v for all $v \in V(G)$.

Conversely, let S^v be an independent 2-dominating set in H^v for all $v \in V(G)$. Define

$$S = \bigcup_{v \in V(G)} S^v,$$

where S^v is an independent 2-dominating set in H^v for all $v \in V(G)$. Then S^v is an independent set in $G \circ H$ for all $v \in V(G)$. Since $V(H^v) \cap V(H^u) = \emptyset$ for $v \neq u$, it follows that

$$S = \bigcup_{v \in V(G)} S^v$$

is an independent set in $G \circ H$. Next, let $y \in V(G \circ H) \setminus S$. Then either $y \in V(G)$ or $y \in V(H^v) \setminus S^v$ for some $v \in V(G)$.

If $y \in V(G)$, then y is adjacent to at least two vertices in S^v since $|S^v| \geq 2$. But $S^v \subseteq S$. Hence, y is adjacent to at least two vertices in S . Therefore, S is a 2-dominating set in $G \circ H$.

If $y \in V(H^v) \setminus S^v$ for some $v \in V(G)$, then y is adjacent to at least two vertices in S^v since S^v is an independent 2-dominating set in H^v . Hence, y is adjacent to at least two vertices in S . Therefore, S is a 2-dominating set in $G \circ H$.

Therefore, by Definition 2.1, S is an independent 2-dominating set in $G \circ H$.

Corollary 3.2 *Let G be a graph of order $m \geq 2$ and H a graph of order $n \geq 2$. Suppose that H admits an independent 2-dominating set. Then $i_2(G \circ H) = m \cdot i_2(H)$.*

Proof: Let S_0 be a minimum independent 2-dominating set in $G \circ H$. By Theorem 3.1,

$$S_0 = \bigcup_{v \in V(G)} S^v,$$

where S^v is an independent 2-dominating set in H^v for all $v \in V(G)$. Thus,

$$\begin{aligned} i_2(G \circ H) &= |S_0|, && \text{by Corollary 2.5(i)} \\ &= \left| \bigcup_{v \in V(G)} S^v \right| \\ &= \sum_{v \in V(G)} |S^v| \\ &= m \cdot |S^v| \\ &\geq m \cdot i_2(H), && \text{by Corollary 2.5(ii).} \end{aligned}$$

Hence, $i_2(G \circ H) \geq m \cdot i_2(H)$.

On the other hand, for each $v \in V(G)$, let S_0^v be a minimum independent 2-dominating set in H^v . Define

$$S = \bigcup_{v \in V(G)} S_0^v.$$

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Then by Theorem 3.1, S is an independent 2-dominating set in $G \circ H$. Thus,

$$\begin{aligned} i_2(G \circ H) &\leq |S|, && \text{by Corollary 2.5(ii)} \\ &= \left| \bigcup_{v \in V(G)} S_0^v \right| \\ &= \sum_{v \in V(G)} |S_0^v| \\ &= m \cdot |S_0^v| \\ &= m \cdot i_2(H), && \text{by Corollary 2.5(i).} \end{aligned}$$

Hence, $i_2(G \circ H) \leq m \cdot i_2(H)$.

Combining the two inequalities, we obtain $i_2(G \circ H) = m \cdot i_2(H)$.

Corollary 3.3 Let G be a graph of order $m \geq 2$ and $\overline{K_n}$ an empty graph of order $n \geq 2$. Then $i_2(G \circ \overline{K_n}) = mn$.

Proof: By Corollary 3.2, $i_2(G \circ \overline{K_n}) = m \cdot i_2(\overline{K_n})$. But $i_2(\overline{K_n}) = n$ by Corollary 2.8. Therefore, $i_2(G \circ \overline{K_n}) = mn$.

Corollary 3.4 Let $c_{n,2}$ be a centipede graph of order $3n$ for $n \geq 2$. Then $i_2(c_{n,2}) = 2n$.

Proof: Note that $c_{n,2} = P_n \circ \overline{K_2}$. Hence,

$$\begin{aligned} i_2(c_{n,2}) &= i_2(P_n \circ \overline{K_2}), \\ &= n \cdot i_2(\overline{K_2}), && \text{by Corollary 3.3} \\ &= n \cdot 2, && \text{by Corollary 2.8} \\ &= 2n. \end{aligned}$$

Therefore, $i_2(c_{n,2}) = 2n$

Corollary 3.5 Let $B(r,r)$ be a bi-star graph of order $2r + 2$ for $r \geq 2$. Then $i_2(B(r,r)) = 2r$.

Proof: Note that $B(r,r) = K_2 \circ \overline{K_r}$, where $r \geq 2$. Thus,

$$\begin{aligned} i_2(B(r,r)) &= i_2(K_2 \circ \overline{K_r}), \\ &= 2 \cdot i_2(\overline{K_r}), && \text{by Corollary 3.3} \\ &= 2 \cdot r && \text{By Corollary 2.8.} \end{aligned}$$

Therefore, $i_2(B(r,r)) = 2r$.

4 INDEPENDENT 2-DOMINATION IN EDGE CORONA OF GRAPHS

This section provides the characterization of the independent 2-dominating set in $G \diamond H$ and the corresponding independent 2-dominating number.

Theorem 4.1 Let G be a connected graph of order $m \geq 2$ and H a graph of order n . Suppose that H admits an independent 2-dominating set. Then a nonempty subset S of $V(G \diamond H)$ is an independent 2-dominating set in $G \diamond H$ if and only if

$$S = \bigcup_{uv \in E(G)} S^{uv},$$

where S^{uv} is an independent 2-dominating set in H^{uv} for all $uv \in E(G)$.

Proof: Let S be an independent 2-dominating set in $G \diamond H$ and $uv \in E(G)$.

We show $S \cap V(G) = \emptyset$. Suppose that there exists $u \in S \cap V(G)$. Since S is independent, it follows that $S \cap V(G)$ is independent. This implies that

$S \cap V(H^{uv}) = \emptyset$ since $uw \in E(G \diamond H)$ for all $w \in V(H^{uv})$. This contradicts the fact that S is a 2-dominating set. Thus, $u \notin S \cap V(G)$. Consequently, $S \cap V(H^{uv}) \neq \emptyset$ and $S \cap V(H^{uv})$ is an independent 2-dominating set in H^{uv} for all $uv \in E(G)$. For each $uv \in E(G)$, let $S^{uv} = S \cap V(H^{uv})$. Accordingly,

$$S = \bigcup_{uv \in E(G)} S^{uv},$$

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where S^{uv} is an independent 2-dominating set in H^{uv} for all $uv \in E(G)$.

Conversely, let S^{uv} be an independent 2-dominating set in H^{uv} for all $uv \in E(G)$. Define

$$S = \bigcup_{uv \in E(G)} S^{uv},$$

where S^{uv} is an independent 2-dominating set in H^{uv} for all $uv \in E(G)$. Then S^{uv} is an independent set in $G \diamond H$ for all $uv \in E(G)$. Since $V(H^{uv})$ and $V(H^{wx})$ are disjoint for all $uv, wx \in E(G)$ with $uv \neq wx$, it follows that

$$S = \bigcup_{uv \in E(G)} S^{uv},$$

is an independent set in $G \diamond H$. Next, let $y \in V(G \diamond H) \setminus S$. Then either $y \in V(G)$ or $y \in V(H^{uv}) \setminus S^{uv}$ for some $uv \in E(G)$.

If $y \in V(G)$, then y is adjacent to at least two vertices in S^{uv} since $|S^{uv}| \geq 2$. But $S^{uv} \subseteq S$. Hence, y is adjacent to at least two vertices in S . Therefore, S is a 2-dominating set in $G \diamond H$.

If $y \in V(H^{uv}) \setminus S^{uv}$ for some $uv \in E(G)$, then y is adjacent to at least two vertices in S^{uv} since S^{uv} is an independent 2-dominating set in H^{uv} . Hence, y is adjacent to at least two vertices in S . Therefore, S is a 2-dominating set in $G \diamond H$.

Therefore, by Definition 2.1, S is an independent 2-dominating set in $G \diamond H$.

Corollary 4.2 Let G be a connected graph of order $m \geq 2$ and H a graph of order n . Suppose that H admits an independent 2-dominating set. Then

$$i_2(G \diamond H) = |E(G)|i_2(H).$$

Proof: Let S_0 be a minimum independent 2-dominating set in $G \diamond H$. By Theorem 4.1,

$$S_0 = \bigcup_{uv \in E(G)} S^{uv},$$

where S^{uv} is an independent 2-dominating set in H^{uv} for all $uv \in E(G)$.

Thus,

$$\begin{aligned} i_2(G \diamond H) &= |S_0| && \text{by Corollary 2.5(i)} \\ &= \left| \bigcup_{uv \in E(G)} S^{uv} \right| \\ &= \sum_{uv \in E(G)} |S^{uv}| \\ &= |E(G)| |S^{uv}| \\ &\geq |E(G)| i_2(H), && \text{by Corollary 2.5(ii)}. \end{aligned}$$

Hence, $i_2(G \diamond H) \geq |E(G)|i_2(H)$.

On the other hand, for each $uv \in E(G)$, let S_0^{uv} be a minimum independent 2-dominating set in H^{uv} . Define

$$S = \bigcup_{uv \in E(G)} S_0^{uv}.$$

Then by Theorem 4.1, S is an independent 2-dominating set in $G \diamond H$. Thus,

$$\begin{aligned} i_2(G \diamond H) &\leq |S| && \text{by Corollary 2.5(ii)} \\ &= \left| \bigcup_{uv \in E(G)} S_0^{uv} \right| \\ &= \sum_{uv \in E(G)} |S_0^{uv}| \\ &= |E(G)| |S_0^{uv}| \\ &= |E(G)| i_2(H), && \text{by Corollary 2.5(i)}. \end{aligned}$$

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Hence, $i_2(G \diamond H) \leq |E(G)|i_2(H)$.

Combining the two inequalities, we obtain $i_2(G \diamond H) = |E(G)|i_2(H)$.

Corollary 4.3 Let G be a connected graph of order $m \geq 2$ and K_n an empty graph of order $n \geq 2$. Then $i_2(G \diamond \overline{K_n}) = |E(G)|n$.

Proof: By Corollary 4.2, $i_2(G \diamond \overline{K_n}) = |E(G)|i_2(\overline{K_n})$. But by Corollary 2.8,

$i_2(\overline{K_n}) = n$. Therefore, $i_2(G \diamond \overline{K_n}) = |E(G)|n$.

5 CONCLUSION

This paper was able to characterize the independent 2-dominating sets in the vertex and edge corona of graphs and derive their independent 2-domination numbers. For further study, the authors recommend to investigate the independent 2-domination number of graphs under some unary operations.

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