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Weakly Convex 2-Domination in the Join of Two Graphs

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ABSTRACT: For a nontrivial graph, a set of vertices of a graph is a weakly convex 2-dominating set if a set is weakly convex and for every vertex in its complement is adjacent to at least two vertices in the set. The weakly convex 2-domination number of a nontrivial graph is the cardinality of a minimum weakly convex 2-dominating set. In this paper, we characterize the weakly convex 2-dominating sets of the join of two graphs and derive the corresponding weakly convex 2-domination number.

KEYWORDS: weakly convex 2-domination, 2-domination, weakly convex domination, domination, join.

1 INTRODUCTION

Weakly convex 2-domination in a graph is a combination of the concept of 2-domination and the notion of a weakly convex set of vertices of a graph. Atibula and Leonida [1] initiated and explored the concept of weakly convex 2-domination. They showed properties and bounds of the weakly convex 2-domination number. Also, they exhibited relationship of this parameter with other domination parameters. In this paper, we characterize the weakly convex 2-domination number of the join of two graphs and derive the corresponding weakly convex 2-domination number.

2 TERMINOLOGY AND NOTATION

Let $G = (V(G), E(G))$ be a graph, where $V(G)$ is the *vertex-set* of G and $E(G)$ is the *edge-set* of G. If $v \in V(G)$, we say that v is a *vertex* of G and if $uv \in E(G)$, we say that uv is an *edge* of G. Two vertices u and v of G are *adjacent* or *neighbors* if uv is an *edge* of G. The *open neighborhood* of v in G, denoted by $N_G(v)$, is a set of neighbors of a vertex v in G. The set $N_G[v] = \{v\} \cup N_G(v)$ is called the *closed neighborhood* of v in G.

If $\deg_G(v) = 0$, that is, v is not adjacent to any other vertex of G; v is an *end-vertex* if $\deg_G(v) = 1$, that is, v is adjacent to exactly one vertex of G; and v is a *support* if v is adjacent to an end-vertex.

A *path* in a graph G is a sequence of distinct vertices $v_0, v_1, ..., v_n$, where there is an edge between each consecutive pair of vertices. The length of a path is the number of edges within it. A $u-v$ path is a path whose initial vertex is u and terminal vertex is v. A graph G is called a *connected* graph if every two vertices in G are joined by a path; otherwise, it is called a *disconnected* graph.

The *degree* of a vertex v of G, denoted by $\deg_G(v)$, is the number of neighbors of v in G, that is, $\deg_G(v) = |N_G(v)|$. The *distance* between vertices u and v, denoted by $d_G(u,v)$, is the length of the shortest u-v path in G. A u-v path of length $d_G(u, v)$ is called *u-v geodesic*. The *diameter* of G, denoted by $\text{diam}(G)$ is defined as $\text{diam}(G) = \max\{d_G(u,v) : u, v \in V(G)\}$.

Let G be a connected graph. A subset C of $V(G)$ is called a *weakly convex* set in G if for every two vertices $u, v \in C$, there exists a u-v geodesic whose vertices belong to C or equivalently, if for every two vertices $u, v \in C$, $d_{\langle C\rangle}(u, v) = d_G(u, v)$.

The *join* of two graphs G and H, denoted by $G + H$, is the graph with vertex-set $V(G+H) = V(G) \cup V(H)$ and edge-set $E(G+H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

A subset S of $V(G)$ is a *dominating set* in G if for every $v \in V(G) \backslash S$, there exists $u \in S$ such that $uv \in E(G)$. The *domination number* of G, denoted by $\gamma(G)$, is the cardinality of a minimum dominating set in G.

Given a connected graph G, a subset C of $V(G)$ is called a *weakly convex dominating set* in G if C is a weakly convex set and a dominating set. The *weakly convex domination number* of G, denoted by $\gamma_{wcon}(G)$, is the cardinality of a minimum weakly convex dominating set in G.

A subset D of $V(G)$ is a 2-*dominating set* in G if every $v \in V(G) \backslash D$, $|D \cap N_G(v)| \geq 2$, that is, v is adjacent to at least two vertices in D. The 2-*domination number* of G, denoted by $\gamma_2(G)$, is the cardinality of a minimum 2-dominating set in G.

Given a nontrivial connected graph G, a subset C of $V(G)$ is called a *weakly convex 2-dominating set* in G if C is a weakly convex set and a 2-dominating set. The *weakly convex 2-domination number* of G, denoted by $\gamma_{2wc}(G)$, is the cardinality of a minimum weakly convex 2-dominating set in G.

Corollary 2.1 *Let G be a nontrivial connected graph.*

(i) *If C_0 is a minimum weakly convex 2-dominating set in G, then* $\gamma_{2wc}(G) = |C_0|$.

(ii) *If C is a weakly convex 2-dominating set in G, then* $\gamma_{2wc}(G) \leq |C|$.

Theorem 2.2 *Let G be a connected graph of order $n \geq 2$. Then* $2 \leq \gamma_{2wc}(G) \leq n$.

3 WEAKLY CONVEX 2-DOMINATION IN THE JOIN OF TWO GRAPHS

In this section we characterize the weakly convex 2-dominating sets in the join of two graphs and derive the weakly convex 2-domination number. Moreover, we characterize graph whose weakly convex 2-domination number equals 2,3 or 4.

Theorem 3.1 *Let G and H be graphs each of order at least 2. Then a subset C of $V(G+H)$ is a weakly convex 2-dominating set in $G+H$ if and only if it one of the following holds:*

(i) $C \cap V(G) = \{u\}$ *is a dominating set in G and* $C \cap V(H) = \{v\}$ *is a dominating set in H.*

(ii) $|C \cap V(G)| = 1$, *and* $|C \cap V(H)| \geq 2$ *and* $C \cap V(H)$ *is a dominating set in H.*

(iii) $|C \cap V(H)| = 1$, *and* $|C \cap V(G)| \geq 2$ *and* $C \cap V(G)$ *is a dominating set in G.*

(iv) $C \subseteq V(G)$ *and C is a weakly convex 2-dominating set in G.*

(v) $C \subseteq V(H)$ *and C is a weakly convex 2-dominating set in H.*

(vi) $|C \cap V(G)| \geq 2$ *and* $|C \cap V(H)| \geq 2$.

Proof: Suppose that C is a weakly convex 2-dominating set in $G+H$. Then C is a weakly convex set in $G+H$ and C is a 2-dominating set in $G+H$. Consider the following cases:

Case 1. $C \cap V(G) \neq \emptyset$ and $C \cap V(H) \neq \emptyset$.

Then $|C \cap V(G)| \geq 1$ and $|C \cap V(H)| \geq 1$. Consider the following subcases: Subcase 1.1. $|C \cap V(G)| = 1$ and $|C \cap V(H)| = 1$. Let $C \cap V(G) = \{u\}$ and $C \cap V(H) = \{v\}$. Then $C = \{u, v\}$ and $uv \in E(G+H)$. We show that $C \cap V(G) = \{u\}$ and $C \cap V(H) = \{v\}$ are dominating sets in G and H, respectively. Let $x \in V(G+H)\backslash C$. Then either $x \in V(G+H)\backslash(C \cap V(G)$ or $x \in V(G+H)\backslash(C \cap V(H)$. If $x \in V(G+H)\backslash(C \cap V(G)$, then $xv \in E(G+H)$. Since C is a 2-dominating set in G+H, x is adjacent to at least 2 vertices in G+H. This implies that there exists $y \in C \cap V(G)$ such that $xy \in E(G)$. Thus, $C \cap V(G) = \{u\}$ is a dominating set in G. Similarly, if $x \in V(G+H)\backslash(C \cap V(H)$, then $C \cap V(H) = \{v\}$ is a dominating set in H. This proves (i).

Subcase 1.2. $|C \cap V(G)| = 1$ and $|C \cap V(H)| > 1$.

Then $|C \cap V(G)| = 1$ and $|C \cap V(H)| \geq 2$. We show that $C \cap V(H)$ is a dominating set in H. Let $C \cap V(G) = \{u\}$ and let $w \in V(H)\backslash(C \cap V(H))$. Then $C = \{u\} \cup (C \cap V(H)$. Suppose, on the contrary, that $C \cap V(H)$ is not a dominating set in H. Then w is not dominated by any vertex in $C \cap V(H)$. Note that $uw \in E(G+H)$. This implies that w is dominated by only one vertex in C. This contradicts the fact that C is a 2-dominating set in $G+H$. Therefore, $C \cap V(H)$ is a dominating set in H. This proves (ii).

Subcase 1.3. $|C \cap V(G)| > 1$ and $|C \cap V(H)| = 1$.

Then $|C \cap V(G)| \geq 2$ and $|C \cap V(H)| = 1$. By similar argument in Subcase 1.2, it can be shown that $C \cap V(G)$ is a dominating set in G. This proves (iii).

Subcase 1.4. $|C \cap V(G)| > 1$ and $|C \cap V(H)| > 1$.

Then $|C \cap V(G)| \geq 2$ and $|C \cap V(H)| \geq 2$. This proves (vi). Case 2. $C \cap V(G) = \emptyset$ or $C \cap V(H) = \emptyset$.

Then either $C \subseteq V(G)$ or $C \subseteq V(H)$. Suppose that $C \subseteq V(G)$. By hypothesis, C is a weakly convex 2-dominating set in $G+H$. Since $C \subseteq V(G)$, it follows that C is a weakly convex 2-dominating set in G. Similarly, if $C \subseteq V(H)$, C is a weakly convex 2-dominating set in H. This proves (iv) and v.

For the converse, firstly, assume that (i) holds, that is, $C \cap V(G) = \{u\}$ is a dominating set in G and $C \cap V(H) = \{v\}$ is a dominating set in H. We show that C is a weakly convex 2-dominating set in $G+H$. Let $C = \{u, v\}$. Then $uv \in E(G+H)$. This means that $\text{diam}_{G+H}(\langle C\rangle = 1$. Thus, C is a weakly convex set in $G+H$. Next, let $x \in V(G+H)\backslash C$. Then either $x \in V(G)\backslash(C \cap V(G))$ or $x \in V(H)\backslash(C \cap V(H))$. If $x \in V(G)\backslash(C \cap V(G))$, then $xu \in E(G) \subseteq V(G+H)$. But $xv \in E(G+H)$, where $v \in C$. This means that x is adjacent to at least two vertices in C. Hence, C is a 2-dominating set in G+H. Therefore, C is a weakly convex 2-dominating set in G+H. Similarly, if $x \in V(H)\backslash(C \cap V(H))$, then C is a weakly convex 2-dominating set in G+H. Secondly, assume that (ii) holds, that is, $|C \cap V(G)| = 1$, and $|C \cap V(H)| \geq 2$ and $C \cap V(H)$ is a dominating set in H. We show that C is a weakly convex 2-dominating set in $G+H$. Let $C \cap V(G) = \{u\}$. Then $C = \{u\} \cup (C \cap V(H))$. Thus, $\langle C\rangle = \langle\{u\}\rangle + \langle C \cap V(H)\rangle$ and $\text{diam}_{G+H}(\langle C\rangle) \leq 2$. Hence, C is a weakly convex set in $G+H$. Next, let $y \in V(G+H)\backslash C$. Then either $y \in V(G)\backslash(C \cap V(G))$ or $y \in V(H)\backslash(C \cap V(H))$.

Case 1. $y \in V(G)\backslash(C \cap V(G))$.

Then $y \in V(G)\backslash\{u\}$. Since $C \cap V(H)| \geq 2$ and by the definition of the join $G+H$, there exists at least two vertices in $C \cap V(H) \subseteq C$ that dominates y. Thus, C is a 2-dominating set in $G+H$. Hence, C is a weakly convex 2-dominating set in $G+H$.

case 2. $y \in V(H)\backslash(C \cap V(H))$.

Since $C \cap V(H)$ is a dominating set in H, y is dominated by at least one vertex in $C \cap V(H)$, that is, y is dominated by at least one vertex in C as $C \cap V(H) \subseteq C$. But by the definition of the join $G+H$, $yu \in E(G+H)$, where $\{u\} = C \cap V(G) \subseteq C$. Thus, y is dominated by at least 2 vertices in

C. Hence, C is a 2-dominating set in $G+H$. Therefore, C is a weakly convex 2-dominating set in $G+H$.

Thirdly, assume that (iii) holds, that is, $|C \cap V(H)| = 1$, and $|C \cap V(G)| \geq 2$ and $C \cap V(G)$ is a dominating set in G. then this case is similar to Case (ii). Hence, Applying similar arguments in (iii), it can be shown that C is a weakly convex 2-dominating set in $G+H$.

Fourthly, assume that (iv) holds, that is, $C \subseteq V(G)$ and C is a weakly convex 2-dominating set in G. But $V(G) \subseteq V(G+H)$. Hence $C \subseteq V(G+H)$ and therefore, C is a weakly convex 2-dominating set in $G+G$. Similarly, if we assume that (v) holds, that is $C \subseteq V(H)$ and C is a weakly convex 2-dominating set in H, then C is a weakly convex 2-dominating set in $G+H$. Lastly, assume that (vi) holds, that is, $|C \cap V(G)| \geq 2$ and $|C \cap V(H)| \geq 2$. Then $\langle C \rangle = \langle C \cap V(G) \rangle + \langle C \cap V(H) \rangle$ and $\text{diam}_{G+H}(\langle C \rangle) \leq$

2. Hence, C is a weakly convex set in $G+H$. Next, let $y \in V(G+H)\backslash C$. Then either $y \in V(G)\backslash C$ or $y \in V(H)\backslash C$. If $y \in V(G)\backslash C$, then by the definition of the join $G+H$ there exist at least 2 vertices in $C \cap V(H) \subseteq C$ that dominates y. Thus, C is a 2-dominating set in $G+H$. Similarly, if $y \in V(H)\backslash C$, then C is a 2-dominating set in $G+H$. Therefore, C is a weakly convex 2-dominating set in $G+H$.

Corollary 3.2 *Let G and H be graphs each of order at least* 2. *Then*

$2 \leq \gamma_{2wc}(G+H) \leq 4$.

Proof: Let $C = \{a, b, x, y\}$, where $a, b \in C \cap V(G)$ and $x, y \in C \cap V(H)$. This implies that $|C \cap V(G)| = 2$ and $|C \cap V(H)| = 2$. Thus, by Theorem, 3.1, C is a weakly convex 2-dominating set in $G+H$. Applying Corollary 2.1(ii), we get $\gamma_{2wc}(G+H) \leq |C| = 4$. The join $G+H$ is a graph of order at least 4. By Theorem 2.2, $2 \leq \gamma_{2wc}(G+H)$. Therefore, $2 \leq \gamma_{2wc}(G+H) \leq 4$.

Corollary 3.3 *Let G and H be graphs each of order at least* 2. *Then*
$\gamma_{2wc}(G+H) = 2$ *if and only if one of the following holds:*

(i) $\gamma(G) = 1$ *and* $\gamma(H) = 1$.

(ii) $\gamma_{2wc}(G) = 2$.

(iii) $\gamma_{2wc}(H) = 2$.

Proof: Suppose that $\gamma_{2wc}(G+H) = 2$. Let C be a minimum weakly convex 2-dominating set in $G+H$. By Theorem 3.1, we have the following cases:

Case 1. By Theorem 3.1(i), $|C \cap V(G)| = 1$ and $|C \cap V(H)| = 1$, where $C \cap V(G)$ is a dominating set in G and $C \cap V(H)$ is a dominating set in H. Clearly, $C \cap V(G)$ is a minimum dominating set in G and $C \cap V(H)$ is a minimum dominating set in H. Hence, $\gamma(G) = 1$ and $\gamma(H) = 1$. This proves (i).

Case 2. By Theorem 3.1(iv), $C \subseteq V(G)$ and C is a weakly convex 2-dominating set in G. Thus, C is a minimum weakly convex 2-dominating set in G by assumption. Hence, $\gamma_{2wc}(G) = 2$. This proves (ii).

Case 3. $C \subseteq V(H)$ and C is a weakly convex 2-dominating set in H. Thus, C is a minimum weakly convex 2-dominating set in H by assumption. Hence, $\gamma_{2wc}(H) = 2$. This proves (iii).

Conversely, assume that $\gamma(G) = 1$ and $\gamma(H) = 1$. Let $C \cap V(G) = \{u\}$ be a dominating set in G and let $C \cap V(H) = \{v\}$ be a dominating set in H. Set $C = \{u, v\}$. By Theorem 3.1, C is a weakly convex 2-dominating set in $G+H$. Thus, by Theorem 3.1(ii), $\gamma_{2wc}(G+H) \leq |C| = 2$. But by Theorem 2.2, $2 \leq \gamma_{2wc}(G+H)$. Therefore, $\gamma_{2wc}(G+H) = 2$. Next, suppose that $\gamma_{2wc}(G) = 2$. Let C be a minimum weakly convex 2-dominating set in G. Then $|C| = 2$. By Theorem 3.1, is a weakly convex 2-dominating set in $G+H$. By Corollary 2.1(ii), $\gamma_{2wc}(G+H) \leq |C| = 2$. But by Theorem 2.2, $2 \leq \gamma_{2wc}(G+H)$. Therefore, $\gamma_{2wc}(G+H) = 2$. Similarly, if $\gamma_{2wc}(G) = 2$, then $\gamma_{2wc}(G+H) = 2$.

Corollary 3.4 *Let G and H be graphs each of order at least* 2 *such that* $\gamma_{2wc}(G+H) \neq 2$. *Then* $\gamma_{2wc}(G+H) = 3$ *if and only if one of the following holds:*

(i) $\gamma(G) = 2$.

(ii) $\gamma(H) = 2$.

(iii) $\gamma_{2wc}(G) = 3$.

(iv) $\gamma_{2wc}(H) = 3$.

Proof: Let G and H be graphs each of order at least 2. Suppose that $\gamma_{2wc}(G+H) \neq 2$. Then $\gamma_{2wc}(G+H) \geq 3$, which

implies that $\gamma_{2wc}(G) \geq 3$ or $\gamma_{2wc}(H) \geq 3$. Assume that $\gamma_{2wc}(G+H) = 3$. Let C be a minimum weakly convex 2-dominating set in $G+H$. Then $|C| = 3$. By Theorem 3.1, we have the following cases:
Case 1. By Theorem 3.1(iv), $C \subseteq V(G)$ and C is a weakly convex 2-dominating set in G. Thus, by Corollary 2.1(ii), $\gamma_{2wc}(G) \leq |C| = 3$. But $\gamma_{2wc}(G) \geq 3$. Hence, $\gamma_{2wc}(G) = 3$. this proves (iii).
Case 2. By Theorem 3.1(v), $C \subseteq V(H)$ and C is a weakly convex 2-dominating set in H. By similar arguments in Case 1, we have $\gamma_{2wc}(H) = 3$. This proves (iv).
Case 3. By Theorem 3.1(ii), $|C \cap V(G)| = 1$ and $C \cap V(H)$ is a dominating set in H. Then $|C \cap V(H)| \geq 2$. Thus $\gamma(H) \leq |C \cap V(H)| = 2$. But $\gamma(H) \geq 2$. Hence, $\gamma(H) = 2$. This proves (ii)
Case 4. By Theorem 3.1(iii), $|C \cap V(H)| = 1$ and $C \cap V(G)$ is a dominating set in G. By similar arguments in Case 3, we have $\gamma(G) = 2$. This proves (i). For the converse, assume first that $\gamma(G) = 2$. Let C_0 be a minimum dominating set in G and let $v \in V(H)$. Set $C = C_0 \cup \{v\}$. Then $|C| = 1+2 = 3$. By Theorem 3.1(ii), C is a weakly convex 2-dominating set in $G+H$. Thus, by Corollary 2.1(ii), $\gamma_{2wc}(G+H) \leq |C| = 3$. But $\gamma_{2wc}(G+H) \neq 2$ implies that $\gamma_{2wc}(G+H) \geq 3$. Hence, $\gamma_{2wc}(G+H) = 3$. Similarly, if $\gamma(H) = 2$, then $\gamma_{2wc}(G+H) = 3$. Next, suppose that $\gamma_{2wc}(G) = 3$. Let $C \subseteq V(G)$ and C is a minimum weakly convex 2-dominating set in G. Then by Theorem 3.1(iv), C is a weakly convex 2-dominating set in $G+H$. By Corollary 2.1(ii), $\gamma_{2wc}(G+H) \leq |C| = 3$. Since $\gamma_{2wc}(G+H) \neq 2$, it follows that $\gamma_{2wc}(G+H) \geq 3$. Therefore, $\gamma_{2wc}(G+H) = 3$. Similarly, if $\gamma_{2wc}(H) = 3$, then $\gamma_{2wc}(G+H) = 3$.

Corollary 3.5 *Let G and H be graphs each of order at least* 2 *such that* $\gamma_{2wc}(G+H) \neq 2, 3$. *Then* $\gamma_{2wc}(G+H) = 4$ *if and only if one of the following holds:*

(i) $\gamma(G) = 3$.

(ii) $\gamma(G) = 3$.

(iii) $\gamma_{2wc}(G) = 4$.

(iv) $\gamma_{2wc}(H) = 4$.

(v) $|C \cap V(G)| = 2$ *and* $|C \cap V(H)| = 2$.

Proof: Let G and H be graphs each of order at least 2. Suppose that $\gamma_{2wc}(G+H) \neq 2, 3$. Then $\gamma_{2wc}(G+H) \geq 4$, which implies that $\gamma_{2wc}(G) \geq 4$ or $\gamma_{2wc}(H) \geq 4$. Assume that $\gamma_{2wc}(G+H) = 4$. Let C be a minimum weakly convex 2-dominating set in $G+H$. Then $|C| = 4$. By Theorem 3.1, we have the following cases:
Case 1. By Theorem 3.1(iv), $C \subseteq V(G)$ and C is a weakly convex 2-dominating set in G. Thus, by Corollary 2.1(ii), $\gamma_{2wc}(G) \leq |C| = 4$. But $\gamma_{2wc}(G) \geq 4$. Hence, $\gamma_{2wc}(G) = 4$. this proves (iii).
Case 2. By Theorem 3.1(v), $C \subseteq V(H)$ and C is a weakly convex 2-dominating set in H. By similar arguments in Case 1, we have $\gamma_{2wc}(H) = 4$. This proves (iv).
Case 3. By Theorem 3.1(ii), $|C \cap V(G)| = 1$ and $C \cap V(H)$ is a dominating set in H. Then $|C \cap V(H)| \geq 3$. Thus, $\gamma(H) \leq |C \cap V(H)| = 3$. Since $\gamma_{2wc}(G+H) \neq 2, 3$, by Corollary 3.5, $\gamma(H) \neq 2$, that is, $\gamma(H) \geq 3$. Hence, $\gamma(H) = 3$. This proves (ii)
Case 4. By Theorem 3.1(iii), $|C \cap V(H)| = 1$ and $C \cap V(G)$ is a dominating set in G. By similar arguments in Case 3, we have $\gamma(G) = 3$. This proves (i). Case 4. By Theorem 3.1(vi), $|C \cap V(G)| = 2$ and $|C \cap V(H)| = 2$. This proves (vi).
For the converse, assume first that $\gamma(G) = 3$. Let C_0 be a minimum dominating set in G and let $v \in V(H)$. Set $C = C_0 \cup \{v\}$. Then $|C| = 1+3 = 4$. By Theorem 3.1(ii), C is a weakly convex 2-dominating set in $G+H$. Thus, by Corollary 2.1(ii), $\gamma_{2wc}(G+H) \leq |C| = 4$. But $\gamma_{2wc}(G+H) \neq 2, 3$ implies that $\gamma_{2wc}(G+H) \geq 4$. Hence, $\gamma_{2wc}(G+H) = 4$. Similarly, if $\gamma(H) = 3$, then $\gamma_{2wc}(G+H) = 4$. Next, suppose that $\gamma_{2wc}(G) = 4$. Let $C \subseteq V(G)$ and C is a minimum weakly convex 2-dominating set in G. Then by Theorem 3.1(iv), C is a weakly convex 2-dominating set in $G+H$. By Corollary 2.1(ii), $\gamma_{2wc}(G+H) \leq |C| = 4$. Since $\gamma_{2wc}(G+H) \neq 2, 3$, it follows that $\gamma_{2wc}(G+H) \geq 4$. Therefore, $\gamma_{2wc}(G+H) = 4$. Similarly, if $\gamma_{2wc}(H) = 4$, then $\gamma_{2wc}(G+H) = 4$. Lastly, Suppose that $|C \cap V(G)| = 2$ and $|C \cap V(H)| = 2$. By Theorem 3.1(vi), C is a weakly convex 2-dominating set in $G+H$.Thus, by Corollary 2.1(ii), $\gamma_{2wc}(G+H) \leq |C| = 4$. Since $\gamma_{2wc}(G+H) \neq 2, 3$, it follows that $\gamma_{2wc}(G+H) \geq 4$. Therefore, $\gamma_{2wc}(G+H) = 4$.

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